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ABSTRACT

A classic problem in the study of the physics of sound in fluids is that of the finite amplitude plane wave propagating in a viscous, unbounded medium. Though the solution to this problem is well known, for a given range of parameters, it would be desirable to develop techniques for solution of the problem which are not similarly limited. Through the application of the technique of parametric differentiation, this goal is realizable. This extension method transforms the governing non-linear differential equation to a linear equation in what may be termed parameter space; the equation is solved and a quadrature recovers the dependent variable solution. Parametric differentiation is conceptually straightforward and has been applied in the past to a wide variety of non-linear equations. The method has been applied to the equation which describes finite amplitude plane wave propagation in a viscous fluid in order to compare the results to the predictions of a known perturbation solution; the ultimate objective being to utilize the technique for exact solution of the corresponding spherical and cylindrical wave problems. The first step in this process has been achieved; that is, numerical solutions for the plane wave case have been generated for a range of values of the various viscous and non-linear parameters which are consistent with results obtained analytically. Further it is shown that solutions generated through the application of parametric differentiation may in fact have greater validity for certain ranges of viscous and non-linear parameters.

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NOTATION

A	the coefficient of the first-order term in an assumed liquid pressure-density relation
A_1, A_2, A_3, A_4	- the variable coefficients of g_{xx}, g_{tt}, g_x, and g_{xxt} respectively
a, c	arbitrary coefficients in a generalized boundary condition
α	small signal attenuation coefficient $= \nu bk^2/2C_o$ for liquids, $= \nu[\Psi + (\gamma-1)/Pr]k^2/2C_o$ for gases
B	the coefficient of the second-order term in an assumed liquid pressure-density relation
b	a viscosity number for liquids which represents shear viscosity and a phenomonological bulk viscosity
β	the parameter of non-linearity $= 1 + B/2A$ for liquids, $= (\gamma+1)/2$ for gases
COO, CO1, etc.	- coefficients of the points of the finite-difference cube
C_o	small signal sound speed
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
Δ	interval or change in the variable which it precedes
ε	acoustic Mach number
ε_n	Neumann factor

η, η' coefficients of shear and bulk viscosity respectively

F the variable inhomogeneous term of the parameter space equation

g parameter space variable = $\partial\xi/\partial\psi$

g_x, g_{xx}, g_{xxt}, ξ_{xx}, etc. - notation for $\partial g/\partial x$, $\partial^2 g/\partial x^2$, $\partial^3 g/\partial x^2 \partial t$, $\partial^2\xi/\partial x^2$, etc.

g_i, ξ_i values of these variables for the ith value of ψ, ψ_i

$g_{i,j}$ indicial notation for the two-dimensional solution grid

Γ an indicator of the strength of the non-linearity relative to dissipation = $\beta\varepsilon k/\alpha$

γ ratio of specific heats = C_p/C_v

I_n nth order modified Bessel function of the first kind

J_n nth order Bessel function of the first kind

k acoustic wave number

κ $\beta\varepsilon k$ product

L[ ] a general linear operator

λ wavelength of acoustic signal, also used as an arbitrary real number in VonNeumann stability test

m, n indicial notation as in $g_{m,n}$, corresponds to $g_{i,j}$

N[ ] a general non-linear operator

ν	kinematic viscosity = η/ρ_o
Ξ_o	particle displacement amplitude = U_o/ω
ξ, ξ_o	particle displacement; subscript indicates the base solution, i=0
ξ_f	an expression for a calculated particle displacement for output
ω	angular frequency
P, Q	constants in the fluid equation of state, for gases $P = p_o$, $Q = 0$; for liquids, $P = \rho_o c_o{}^2/\gamma$, $Q = P - p_o$, and γ is determined empirically
p	instantaneous pressure
p_o	ambient pressure
Pr	Prandtl number
ϕ, ϕ_o	an arbitrary dependent variable, subscript indicates the base solution
ρ	instantaneous density
ρ_o	ambient density
s	condensation = $\rho-\rho_o/\rho_o$
σ	a nondimensional distance = $\beta\varepsilon kx$
T	shock thickness

t, t_c	temporal coordinate, characteristic time
θ	an arbitrary complex number in VonNeumann stability test
θ_{B_i}	boundary values of ϕ
u, u_o	particle velocity, subscript indicates base solution
U_o	particle velocity amplitude
$\overline{X}$	discontinuity distance = $1/\beta\varepsilon k$
$\vec{X}$, $\vec{X}_B$	a generalized spatial vector, the subscript indicates the boundary
x, x_c	spatial coordinate, characteristic length
Ψ	viscosity number
ψ, ψ_o	parameter of interest = $\nu b/C_o\lambda$, subscript indicates base value
V	u/U_o
y	a timelike variable in the Burgers equation solution
ζ	a non-physical variable used to transform the Burgers equation to a heat equation

I. INTRODUCTION

The propagation of acoustic waves in fluids and certain crystalline substances has been the subject of long and exhaustive study. Due to the inherent simplifications which arise in the examination of plane wave propagation, this problem has received the closest scrutiny. In the general linear theory of sound propagation, the assumption must be made that particle velocity amplitudes are infinitesimal, allowing the reduction of the problem to the classic wave equation. However, in many of today's acoustic systems this assumption is no longer valid, and solution of the fully non-linear wave propagation problem is required for analysis and prediction for practical systems. It has long been recognized that sound waves whose particle velocity amplitudes are not infinitesimal have phase velocities whose magnitudes change with the local particle velocity, a result which is not predicted by the linear theory. This phenomenom received some attention from the philosophers of the nineteenth century, and with the exception of the efforts of two researchers of the pre-war decade was not treated until the last two decades at which time significant interest was generated because of the increasing sophistication and power levels of acoustic systems.

The solutions of the 1930's form the basis of this research. Fubini [1] derived an implicit solution to the plane-wave problem in an inviscid fluid and reduced it to an explicit

solution for low Mach number. Fay [2], the other researcher of the 30's, solved the same equation with viscous effects included. Blackstock [3] noted that though both solutions were entirely correct, they were restricted in their spatial regions of applicability. Applying weak-shock theory, he demonstrated the manner in which the two solutions are related and developed a single function which describes the solution for all space in the inviscid case. Keck and Beyer [4] performed a perturbation analysis which, for a specific range of parameters, described propagation in the viscous case. Blackstock [5], using approximations which he demonstrated [6] to be equivalent to those used by Keck and Beyer, was able to reduce the governing differential equation for plane wave propagation in a thermoviscous fluid to the Burgers equation for a boundary value problem, which yielded an analytic solution for all space and time. Propagation in relaxing fluids has received moderate attention [7,8], but has generally been avoided. Blackstock [9] has derived analogous inviscid solutions for cylindrical and spherical waves, but as fate would have it, an analytic solution for cylindrical and spherical waves in a viscous fluid has not yet been obtained due to the fact that there exists no apparent simple transform, such as that utilized in the plane wave case, to obtain the Burgers equation.

There are various parameters which describe the proper-

ties of the medium and two which relate to the source excitation, and taken together they give one an indication of the relative strengths of non-linear effects and viscous (dissipative) effects. The parameters of the medium are ν, the kinematic viscosity; Ψ, the viscosity number; γ, the ratio of specific heats; Pr, the Prandtl number; β, the parameter of non-linearity; and C_o, the small signal sound speed. The expression $(\Psi + \frac{\gamma-1}{Pr})$ is a collection of thermal and viscous coefficients which together with kinematic viscosity relate the thermoviscous nature of gases. In liquids, such an expression is not readily determined and is generally replaced by a single term, b. β is equal to $(\gamma + 1)/2$ for perfect gases and to $1 + B/2A$ for fluids of arbitrary equation of state where A is the coefficient of the first order term and B/2 is the coefficient of the second order term in an assumed pressure-density relation. Analogous parameters for dielectric crystals allow them to be treated in a manner similar to that for thermoviscous fluids [10]. The two parameters related to the source excitation are U_o, the peak particle velocity, and ω, the angular frequency of the source. Much of the previous effort in this field has made use of a parameter $\Gamma = \beta\varepsilon k/\alpha$ to express the relative strength of non-linearity and viscosity, where $\varepsilon = U_o/C_o$, the acoustic Mach number; $k = \omega/C_o$, the wave number; and $\alpha = \nu b k^2/2C_o$, the small signal attenuation coefficient. $\Gamma \sim 1$ may be considered as the borderline for the inception of important non-linear effects [11].

Another important parameter is the discontinuity distance

$\overline{X} = (\beta\varepsilon k)^{-1}$; at this point, in inviscid propagation, the slope of the particle velocity waveform becomes negatively infinite due to non-linear generation of harmonics, that is, $\overline{X}$ indicates the point at which a shock first forms. Shocks were first predicted by the inviscid theory, and the discontinuity distance which has become a reference point in most non-linear theory, was predicted by the Fubini solution, which, in fact, is not valid beyond the discontinuity distance. Shocks form because the finite amplitude of the particle velocity causes the compressive portion of the waveform to 'catch up' to the rarefactive portion of the waveform; because a multi-valued waveform is a physical impossibility, a shock must form unless dissipation reduces the wave amplitude sufficiently. Figures 1a and 1b show graphically such a waveform. The utility of Γ has been extended by Fenlon [12] to indicate whether and where a shock will form in the viscous case; for plane waves $\Gamma > 4.5$ indicates that shock formation will occur, that is, non-linear effects will be sufficiently strong, so that even with dissipation, shocks will form. Shock formation marks the end of the first of three identifiable zones of propagation of the finite amplitude wave; this is a region of strong non-linear effects. The second region of propagation is that in which non-linear effects initially dominate, but gradually taper off as absorption reduces the magnitude of the fundamental and even more rapidly the magnitudes of the non-linearly generated

harmonics. The limit of this region is reached when the rate of decay of the fundamental due to small signal attenuation is matched by the rate of decay due to the generation of harmonics, that is, when the particle velocity amplitudes are once again infinitesimal.

The aforementioned solutions may be discussed with respect to their relations to the parameters of the problem. The Fubini solution predicts shock formation at the discontinuity distance, but the solution is no longer convergent beyond the discontinuity distance. Fay produced a solution for the most stable waveform, a sawtooth, to which the finite amplitude wave generates beyond about 3 discontinuity distances. The Fay solution contained Γ as a parameter, but did not reduce to Fubini for $\Gamma \to \infty$; Blackstock's bridging function accomplished this. Keck and Beyer's perturbation solution is valid for very weak non-linear effects (relative to dissipative effects). Because of the limited number of terms generated, it is valid for $\Gamma = O(1)$ or less. Finally, Blackstock's Burgers equation solution is valid for all Γ. Unfortunately the series which represents the steady state solution is very slowly convergent, limiting the usefulness of the solution. Finally, it should be noted that an inherent restriction on the size of the Mach number is assumed implicitly or explicitly in all cases, with a maximum value of $\varepsilon = .1$. Such a value would be stretching the applicability of certain approximations which

have been made in each of the above cases.

In order to dispense with the aforementioned problems with the two viscous solutions, and specifically, to essentially eliminate the approximations which were made therein, another technique must be used which returns to the basic partial differential equation and solves it without approximation. Parametric differentiation is such a technique. Given a non-linear differential equation in which there exists a parameter, one need only assume that the solution is continuous in that parameter to a limiting value for which a known solution exists. This technique has been applied in the past by Rappert and Landahl [13] to non-linear flow problems and by Harris [14] to aerodynamic sound produced by a rotating cylinder in a viscous medium. VanDyke [15] has linked parametric differentiation to work he has performed on the extension of perturbation series solutions. The application of parametric differentiation in this case to a problem whose approximate solution is known will (1) shed some additional light on the physics of the problem, and (2) demonstrate the applicability and utility of a new technique for use in studies of the propagation of finite amplitude cylindrical and spherical waves in a viscous medium.

II. THE EQUATION OF MOTION

A plane wave propagating isentropically in a homogeneous viscous fluid may be completely described by the continuity equation, the Navier-Stokes equation for irrotational flow and an equation of state. The appropriate equations are:

1) Continuity $\rho = (1 + \partial\xi/\partial x)\rho_o$ (2.1)

2) Navier-Stokes $\rho_o \partial^2\xi/\partial t^2 = -\partial p/\partial x + (\frac{4}{3}\eta + \eta')\partial^3\xi/\partial x^2\partial t$ (2.2)

3) State $p = P(\rho/\rho_o)^\gamma - Q$ [4] (2.3)

where ρ is the density, ξ is the particle displacement, p is the pressure, η is the shear viscosity coefficient, η' is the bulk viscosity coefficient, and P and Q are constants dependent on the fluid considered. The subscripted parameters denote rest values of the respective quantities. For ideal gases, $P = p_o$, $Q = 0$, and γ is the ratio of specific heats, C_p/C_v. For liquids, $P = \rho_o C_o^2/\gamma$, where C_o is the small signal sound speed, $Q = P - p_o$, and γ must be determined empirically with the assumed equation of state above. The equation of state for liquids is often written in the form:

$$p = p_o + A(\frac{\rho-\rho_o}{\rho_o}) + B/2(\frac{\rho-\rho_o}{\rho_o})^2 + O\left((\frac{\rho-\rho_o}{\rho_o})^3\right)\ldots \qquad (2.4)$$

with $\frac{\rho-\rho_o}{\rho_o} = s$, the condensation. For small values of s, this

equivalent to the equation of state shown (2.3). This can be shown by writing

$$p = P\left(1 + \frac{\rho-\rho_o}{\rho_o}\right)^{\gamma} - Q \tag{2.5}$$

and performing a binomial expansion for $\left|\frac{\rho-\rho_o}{\rho_o}\right| < 1$, the result being

$$p = P + \gamma P\left(\frac{\rho-\rho_o}{\rho_o}\right) + \frac{\gamma(\gamma-1)}{2} P\left(\frac{\rho-\rho_o}{\rho_o}\right)^2 + \ldots - Q \tag{2.6}$$

Then, with $P - Q = p_o$, A is equivalent to γP, and B to $\gamma(\gamma-1)P$ and the ratio $B/A = \gamma-1$. When one uses this form the implicit assumption is made that terms of $O\left(\left(\frac{\rho-\rho_o}{\rho_o}\right)^3\right)$ and greater are not important. For any fluid of low compressibility this is clearly valid for small acoustic Mach number. (Note that $\rho/\rho_o = (1 + \frac{\partial\xi}{\partial x})^{-1}$ and $\frac{\partial\xi}{\partial x} \sim \varepsilon$).

Returning to Equations 2.1-3, p and ρ may be eliminated to yield a single equation in the particle displacement. From Equation 2.3

$$\frac{\partial p}{\partial x} = \frac{P\gamma}{\rho_o}\left(\frac{\rho}{\rho_o}\right)^{\gamma-1}\frac{\partial \rho}{\partial x} \tag{2.7}$$

Substituting Equation 2.1 and its x derivative, $\frac{\partial\rho}{\partial x} = -\rho_o\frac{\partial^2\xi}{\partial x^2} \cdot (1 + \frac{\partial\xi}{\partial x})^{-2}$, into Equation 2.7, one obtains,

$$\frac{\partial p}{\partial x} = -P\gamma\left(1 + \frac{\partial \xi}{\partial x}\right)^{-\gamma-1} \frac{\partial^2 \xi}{\partial x^2} \tag{2.8}$$

Substituting in the Equation 2.2, rearranging terms, and inserting $P = \rho_o C_o^{\,2}/\gamma$, one obtains the following equation:

$$\left(1 + \frac{\partial \xi}{\partial x}\right)^{\gamma+1}\left(\frac{\partial^2 \xi}{\partial t^2} - \nu b\,\frac{\partial^3 \xi}{\partial x^2 \partial t}\right) = C_o^2\,\frac{\partial^2 \xi}{\partial x^2} \tag{2.9}$$

with $\nu = \eta/\rho_o$, $b = \frac{4}{3} + \frac{\eta'}{\eta}$. At this point one may leave the equation as it stands and apply parametric differentiation directly or make one additional simplification which resorts to the assumption that the displacements though finite are still not large enough to require the retention of cubic terms in particle displacement. Since previous effort has been directed along the second path and since this previous work is the point for comparison to determine the validity of the approach, this additional simplification is made. $\left(1 + \frac{\partial \xi}{\partial x}\right)^{-\gamma-1}$ is expanded binomially to yield:

$$\left(1 + \frac{\partial \xi}{\partial x}\right)^{-\gamma-1} = 1 - (\gamma+1)\frac{\partial \xi}{\partial x} + (\gamma+1)\frac{(\gamma+2)}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 + O\left(\left(\frac{\partial \xi}{\partial x}\right)^3\right)\ldots \tag{2.10}$$

The first two terms only are retained since this factor multiplies the right hand side. The resulting equation is:

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{C_o^2}\frac{\partial^2 \xi}{\partial t^2} + \frac{\nu b}{C_o^2}\frac{\partial^3 \xi}{\partial x^2 \partial t} = (2\beta)\frac{\partial \xi}{\partial x}\frac{\partial^2 \xi}{\partial x^2} \tag{2.11}$$

where $\beta = (\gamma+1)/2$ for gases and $\beta = 1 + B/2A$ for liquids. This is in essence the equation treated by Keck and Beyer [4], along with the boundary condition $\xi(0,t) = -\Xi_0\cos\omega t$, in their perturbation analysis, after terms of $O\left((\alpha/k)^2\right)$ are eliminated. The Keck-Beyer solution is computationally simple, but it is severely limited by the labor required to calculate the terms of the perturbation series. In fact, according to the authors, the perturbation series is probably only convergent for $\beta\varepsilon k(1-e^{-2\alpha x})/4\alpha < 1$, which restriction corresponds, for example, to an acoustic pressure of .125 atmospheres at 1 MHz in water; one could easily argue that this pressure hardly qualifies as finite amplitude,but then if one is able to measure finely enough, any acoustic signal can be considered to be of finite amplitude. Actually, the series has been demonstrated to be everywhere convergent but is still limited in applicability, by truncation, to values of the parameters as defined above [7]. It is to be noted also that $\beta\varepsilon k/4\alpha = \Gamma/4$; thus the Keck-Beyer solution will not allow $\Gamma > 4$, and since $\Gamma > 4.5$ is required for shock formation in the viscous plane wave case, this solution deals with non-shocked waves only.

Blackstock has demonstrated that the governing equation may be reduced to a Burgers equation by applying the same assumption utilized in the perturbation analysis. This assumption, in fact, is the same one which allows one to arrive at an analytic steady state solution of the following problem:

$$\frac{\partial^2\xi}{\partial t^2} - \frac{\nu b\partial^3\xi}{\partial x\partial^2 t} = C_o^2\,\frac{\partial^2\xi}{\partial x^2}\,, \tag{2.12}$$

namely the small signal problem for plane wave propagation in a viscous fluid. This assumption, that $\alpha/k << 1$, is valid for most fluids, with the exception of the most viscous ones, at any reasonable acoustic frequency. In any case, the effect of making this assumption is to allow one to utilize the expression:

$$\frac{\partial\xi}{\partial x} = -\frac{1}{C_o}\frac{\partial\xi}{\partial t}\,, \tag{2.13}$$

in reducing the full equation to a Burgers equation. This expression is, of course, the differential equation which describes the propagation of a small amplitude wave in a lossless medium. The reduction procedure follows. Starting with Equation 2.11 (an approximate one which has been obtained by assuming a small Mach number), one first substitutes $\xi_x = -\frac{1}{C_o}\xi_t$ on the right hand side to obtain:

$$\xi_{xx} - \frac{1}{C_o{}^2}\xi_{tt} + \frac{\nu b}{C_o{}^2}\xi_{xxt} = -\frac{2\beta}{C_o}\xi_t\xi_{xx}\,, \tag{2.14}$$

Then note the following identities from Equation 2.13:

$$\xi_{xt} = -\frac{1}{C_o}\xi_{tt} \qquad \text{(a)} \qquad\qquad C_o\xi_{xtt} = -\xi_{ttt} \qquad \text{(c)}$$

$$C_o\xi_{xxt} = -\xi_{xtt} \qquad \text{(b)} \qquad\qquad C_o^2\xi_{xxt} = -C_o\xi_{xtt} = \xi_{ttt} \qquad \text{(d)}$$

$$\xi_{xx} = -\frac{1}{C_o}\xi_{xt} \quad (e) \qquad \xi_{xx} = \frac{1}{C_o^{\,2}}\xi_{tt} \quad (f) \qquad (2.15a\text{-}f)$$

Substituting (d) into the viscous term of Equation 2.14,

$$\frac{\nu b}{C_o^{\,2}}\xi_{xxt} = \frac{\nu b}{2C_o^{\,2}}(\xi_{xxt} + \xi_{xxt}) = \frac{\nu b}{2C_o^{\,4}}(\xi_{ttt} - C_o\xi_{xtt}) \text{ and}$$

$$\xi_{xx} - \frac{1}{C_o^{\,2}}\xi_{tt} + \frac{\nu b}{2C_o^{\,4}}(\xi_{ttt} - C_o\xi_{xtt}) = -\frac{2\beta}{C_o}\xi_t\xi_{xx} \qquad (2.16)$$

Substituting (e) and (f) into the non-linear term of Equation 2.16,

$$-\frac{2\beta}{C_o}\xi_t\xi_{xx} = \frac{\beta}{C_o}(-\xi_t\xi_{xx} - \xi_t\xi_{xx})$$

$$= \frac{\beta}{C_o}\left(\frac{1}{C_o}\xi_t\xi_{xt} - \frac{1}{C_o^{\,2}}\xi_t\xi_{tt}\right)$$

The equation after substituting and manipulating signs becomes

$$C_o^2\xi_{tt} + \frac{\nu b}{2}(C_o\xi_{xtt} - \xi_{ttt}) - C_o^4\xi_{xx} = \beta C_o(\xi_t\xi_{tt} - C_o\xi_t\xi_{xt}) \qquad (2.17)$$

Performing an operation which is the integral equivalent of differentiating with respect to the operator $\frac{\partial}{\partial t} - C_o\frac{\partial}{\partial x}$, one obtains

$$C_o^2\xi_t - \frac{\nu b}{2}\xi_{tt} + C_o^3\xi_x = \frac{1}{2}\beta C_o\xi_t^{\,2} \qquad (2.18)$$

Differentiating with respect to 't' and substituting $u = \xi_t$,

$$C_o^2 u_t - \frac{\nu b}{2} u_{tt} + C_o^3 u_x = \beta C_o u \, u_t \tag{2.19}$$

Finally, with the variable transform

$$x' = x \qquad t' = t - \frac{x}{C_o} \quad ,$$

one obtains

$$C_o^3 u_{x'} - \beta C_o u \, u_{t'} = \frac{1}{2} \nu b u_{t't'} \quad , \tag{2.20}$$

which is Burgers equation for a boundary value problem. The significant point here is that one is considering an equation which is non-linear, and which includes viscous terms. The approximation $\alpha/k << 1$, is applied two times to reduce the equation to Burgers form, each time to simplify viscous and non-linear terms. This is indeed an acceptable engineering approximation; however, it certainly is desirable to have the ability to solve the equation without resorting to this approximation, particularly for highly viscous fluids and perhaps certain dielectric crystals with large stress coefficients.

As noted by Blackstock [6], the Burgers equation solution has one further limitation which might be considered more

severe. The steady state solution to the Burgers equation is

$$\zeta = \sum_{n=1}^{\infty} \varepsilon_n (-1)^n I_n (\tfrac{1}{2}\Gamma) e^{-n^2\sigma/\Gamma} \cos ny \tag{2.21}$$

where ζ is related to u by,

$$u = U_o \frac{2}{\Gamma} [\log \zeta]_y \tag{2.22}$$

Here, ε_n is the Neumann factor ($\varepsilon_o = 1$, all other $\varepsilon_n = 2$); I_n is the nth order Bessel of imaginary argument; $\sigma = \beta\varepsilon kx$, a non-dimensional space variable; and $y = \omega t'$, a non-dimensional temporal variable. For values of $\Gamma > 50$, the series representation of the solution is very slowly convergent, and therefore difficult to use practically in numerical analysis. Consequently, for $\Gamma > 50$, Blackstock utilized an asymptotic approach to the solution, which in fact is not valid near the origin. To solve the Burgers equation for $\beta\varepsilon kx \ll \Gamma$, he resorts to a perturbation analysis in Γ^{-1}, but recalling $\Gamma = \beta\varepsilon k/\alpha$, this is in some sense similar to the Keck-Beyer approach which utilized ε as the perturbation quantity to solve the same basic equation. Blackstock's solution for small x contains terms of first order in the perturbation parameter; for $\Gamma \gg \sigma$, this should indeed be appropriate, however the coefficients derived contain an infinite series of Bessel functions which may be

slowly convergent themselves. Herein lies another motivation for the parametric differentiation solution, the desire to produce a unified solution applying the same assumptions and having the same limitations throughout.

In any event, there is no simple transform which will reduce either (1) the full equation valid for high Mach number, or (2) cylindrical and spherical finite amplitude wave equations to the Burgers form. Therefore, an alternative approach to the problem is required.

III. PARAMETRIC DIFFERENTIATION

Parametric differentiation is a procedure which allows a non-linear equation which involves a parameter to be solved by transforming the equation to a linear equation in parameter space. This is particularly valuable for cases in which numerical solution will be required because it eliminates the possibility of obtaining multiple roots in solving non-linear difference equations. It is essential to note at this point that some insight into the nature of the solution at the extended range of the parameter of interest would be extremely helpful. If the solution is singular in the parameter, that is, if the nature of the solution changes radically as one approaches the limiting value of the parameter in the known base solution, then the technique is not applicable. Thus, parametric differentiation would never be applicable to the entire class of singular perturbation problems. The method of application follows.

Suppose one is given the general problem:

$$N[\phi(\vec{X},\ t;\ \psi)] = 0 \tag{3.1}$$

$$\phi[(\vec{X}_B,\ t;\ \psi)] = \theta_{B_1}(t;\ \psi) \tag{3.1a}$$

$$\phi_\eta[(\vec{X}_B,\ t;\ \psi)] = \theta_{B_2}(t;\ \psi) \tag{3.1b}$$

$$a\phi[(\vec{X}_B,\ t;\ \psi)] + c\phi_n(\vec{X}_B,\ t;\ \psi) = \theta_{B_3}(t;\psi) \tag{3.1c}$$

where N is a non-linear operator; $\vec{X}$ is a spacial vector coordinate; t is a time coordinate; ψ is the parameter of interest; and the θ_B's are functional values of ϕ, ϕ_n, or $a\phi + c\phi_n$ at $\vec{X}_B$ as appropriate to the problem, n indicating the normal derivative. Initial values could also be prescribed if appropriate. If a solution $\phi_o = \phi(\vec{X},t;\ \psi_o)$ exists at some limiting value of the parameter ψ_o, and it is desired to obtain a solution to the same problem with identical boundary and/or initial values at $\psi_o + \Delta\psi$, then parametric differentiation may be applied. First, differentiate the governing equation and boundary and/or initial conditions with respect to the parameter to yield:

$$L[g(\vec{X},\ t;\ \psi)] = 0 \tag{3.2}$$

$$g(\vec{X}_B,\ t;\ \psi) = g_{B_1}(t;\ \psi) \tag{3.2a}$$

$$g_n(\vec{X}_B,\ t;\ \psi) = g_{B_2}(t;\ \psi) \tag{3.2b}$$

$$ag(\vec{X}_B,\ t;\ \psi) + cg_n(\vec{X}_B,\ t;\ \psi) = g_{B_3}(t;\ \psi) \tag{3.2c}$$

where

$$g(\vec{X},\ t;\ \psi) = \frac{\partial}{\partial\psi}\ (\phi(\vec{X},\ t;\ \psi)) \tag{3.3}$$

One should realize that this procedure cannot be applied if

the parameter appears in the basic equation raised to a power less than one, and if the limiting value of the parameter $\psi_o = 0$.

The operator, L, is a new linear operator obtained through differentiation. The new equation with associated boundary and/or initial conditions may now be solved in parameter space, either analytically or numerically, for successive values of the parameter ψ. As the equation is solved for each value ψ, a quadrature is performed which solves Equation 3.3 and returns a value or functional form which, when applied to the solution for ψ_{i-1}, yields the desired solution at ψ_i. There is no numerical restriction on the range of ψ; however, there certainly may be a physical limit, and some insight is required here to avoid blind application of the method to obtain, for instance, solutions for a range of ψ which is beyond the region of validity of the basic equation. The quadrature may be expressed as follows:

$$\phi_i(\vec{X}, t; \psi_i) = \phi_{i-1}(\vec{X}, t; \psi_{i-1}) + \int_{\psi_{i-1}}^{\psi_i} g d\psi \tag{3.4}$$

For more complicated equations which require numerical solution, this step can be quite complex, depending on the form of the g versus ψ curve. In some cases, trapezoidal integration may be sufficient; in others, higher order integration techniques may be required. It is clear in examining the form of

the equation above, that this is, in essence, a recursion type relation and consequently it demands a known initializing function for i=1.

In this work, the initializing function is the Fubini solution to the problem,

$$(1 + \frac{\partial \xi}{\partial x})^{2\beta} \frac{\partial^2 \xi}{\partial t^2} = C_o^2 \frac{\partial^2 \xi}{\partial x^2} \tag{3.5}$$

with the boundary condition $\xi(0, t) = - \Xi_o \cos\omega t$, this being a duplicate of Equation 2.9, without the dissipative term. Thus, the parameter of interest here will be some non-dimensional quantity which contains ν, and the quadrature will be performed with respect to the non-dimensional quantity. An explicit form of the Fubini solution was given by Keck and Beyer [4] in terms of particle velocity,

$$u(x, t) = 2u_o \sum_{n=1}^{\infty} \frac{J_n(n\kappa x)}{n\kappa x} \sin n(\omega t - kx) \tag{3.6}$$

where J_n is the nth order Bessel function and $\kappa = \beta\epsilon k$. This solution is limited in Mach number ($\frac{u_o}{C_o} << 1$) since a binomial expansion in Mach number was utilized to arrive at this result. An exact implicit solution is available which could be used for analysis at greater Mach numbers,

$$u(x, t) = u_o \sin(\omega t - \frac{\omega x}{C_o} (1 + \frac{\gamma-1}{2} \frac{u}{C_o})^{-(\frac{\gamma+1}{\gamma-1})}) \tag{3.7}$$

However, the numerical problems associated with an iterative solution at many points would be difficult to resolve. This solution predicts the formation of the shock at the point where the velocity profile becomes negatively infinite, that is, where $\partial u/\partial x \to -\infty$. Differentiating Equation 3.7 with respect to x,

$$\frac{\partial u}{\partial x} = U_o \cos(\omega t - \frac{\omega x}{C_o}(1 + \frac{\gamma-1}{2}\frac{u}{C_o})^{-(\frac{\gamma+1}{\gamma-1})})[-\frac{\omega}{C_o}(1 + \frac{\gamma-1}{2}\frac{u}{C_o})^{-(\frac{\gamma+1}{\gamma-1})}$$

$$+ \frac{\gamma+1}{\gamma-1}\frac{\omega x}{C_o}(1 + \frac{\gamma-1}{2}\frac{u}{C_o})^{-(\frac{\gamma+1}{\gamma-1})-1}\frac{\gamma-1}{2C_o}\frac{\partial u}{\partial x}] \tag{3.8}$$

or

$$\frac{\partial u}{\partial x} = -\frac{U_o\omega}{C_o}(1 + \frac{\gamma-1}{2}\frac{u}{C_o})^{-(\frac{\gamma+1}{\gamma-1})}\cos(\quad)/(1 - \frac{\gamma+1}{2}\frac{U_o}{C_o}\frac{\omega x}{C_o}$$

$$\cdot (1 + \frac{\gamma-1}{2}\frac{u}{C_o})^{-(\frac{\gamma+1}{\gamma-1})-1}\cos(\quad)) \tag{3.9}$$

This quantity becomes infinite when

$$x = (1 + \frac{\gamma-1}{2}\frac{u}{C_o})^{\frac{2\gamma}{\gamma-1}}/\frac{\gamma+1}{2}\frac{U_o}{C_o}\frac{\omega}{C_o}\cos(\quad) \tag{3.10}$$

Noting that $\beta = \frac{\gamma+1}{2}$, $\varepsilon = \frac{U_o}{C_o}$, $\frac{\omega}{C_o} = k$, and that the first shock will occur at the head of the waveform where cos( ) = 1, and

also that u = 0 at the shock, the discontinuity distance $\overline{X}$ $= \frac{1}{\beta\varepsilon k}$ is obtained. $\overline{X}$ is the limit of convergence of the series solution stated above. Thus, using the Fubini explicit solution as a starting solution, one is limited to application of the technique of parametric differentiation in the region $0 < x < \overline{X}$ and for Mach numbers which are small compared to unity. It is clear from this discussion that one is fundamentally limited when applying parametric differentiation by the quality of the base solution. It is unfortunate that in this case, the base solution is limited with respect to Mach number; on the other hand, signals of sufficiently large amplitude may no longer be thought of as acoustic waves and may propagate in a different manner.

IV. APPLICATION OF THE METHOD

It is one thing to discuss the use of parametric differentiation in abstract terms, and quite another to execute its application, there being no real 'theory' with respect to how it should be applied. In any case, the first step in the solution technique is similar to that for most other numerical solution procedures, that is, non-dimensionalize the equation. Starting with the reduced equation

$$\xi_{xx} - \frac{1}{C_o^2}\,\xi_{tt} + \frac{\nu b}{C_o^2}\,\xi_{xxt} = 2\beta\xi_x\xi_{xx}\ , \tag{4.1}$$

one introduces characteristic lengths and times. This being a wave propagation problem,it is reasonable to select $x_c = \lambda$ and $t_c = \frac{\lambda}{C_o}$, where λ is the wavelength; then the non-dimensional variables are:

$$\hat{x} = x/x_c = x/\lambda \qquad \hat{t} = t/t_c = tC_o/\lambda \quad \text{and} \quad \hat{\xi} = \xi/x_c = \xi/\lambda$$

Then, Equation 4.1, in terms of dimensionless variables, is:

$$\hat{\xi}_{\hat{x}\hat{x}} - \hat{\xi}_{\hat{t}\hat{t}} + \frac{\nu b}{C_o\lambda}\,\hat{\xi}_{\hat{x}\hat{x}\hat{t}} = 2\beta\hat{\xi}_{\hat{x}}\hat{\xi}_{\hat{x}\hat{x}} \tag{4.2}$$

This equation is as one would expect for a 'good' non-dimensionalization; the coefficients of the first two terms, which in fact should dominate, are of O(1). Alternatively, with the

equation in the form

$$(1 + \xi_x)^{2\beta}(\xi_{tt} - \nu b \xi_{xxt}) = C_o^2 \xi_{xx} \tag{4.3}$$

a similar non-dimensionalization will yield

$$(1 + \hat{\xi}_{\hat{x}})^{2\beta}(\hat{\xi}_{\hat{t}\hat{t}} - \frac{\nu b}{C_o \lambda} \hat{\xi}_{\hat{x}\hat{x}\hat{t}}) = \hat{\xi}_{\hat{x}\hat{x}} \tag{4.4}$$

which demonstrates that the relatively large numerical value of the coefficient of the non-linear terms in 4.2 is simply a result of the expansion performed, and should not be considered improper. The viscosity coefficient appears as $\frac{\nu b}{C_o \lambda}$ which is like an inverse Reynold's number. However, because C_o is a propagation speed and not a particle velocity, this is not a classic acoustic Reynold's number. In any case, this quantity, which will be termed 'ψ', is the parameter of interest, on which the quadrature will be performed. The solution for ψ=0 is known; it is the Fubini solution. Then one has

$$u_o(X, t; \psi_o) = 2U_o \sum_{n=1}^{\infty} \frac{J_n(n\kappa x)}{n\kappa x} \sin n(\omega t - kx) \tag{4.5a}$$

or, in terms of particle displacement

$$\xi_o(X, t; \psi_o) = -2\frac{U_o}{\omega} \sum_{n=1}^{\infty} \frac{J_n(n\kappa x)}{n^2 \kappa x} \cos n(\omega t - kx) \tag{4.5b}$$

which corresponds to the function ϕ_o in the theoretical devel-

opment of parametric differentiation, and with $\Xi_o = \frac{o}{\omega_o}$.

The next step in the development is to differentiate Equation 4.2 with respect to ψ, yielding

$$g_{xx} - g_{tt} + \psi g_{xxt} + \xi_{xxt} = 2\beta\xi_x g_{xx} + 2\beta\xi_{xx} g_x \tag{4.6}$$

where

$$g = \frac{\partial\xi}{\partial t} \tag{4.7}$$

and the umlaut notation has been dropped from the non-dimensional variables. One would immediately look at this equation and express horror at what has been done; there seems to be added complexity rather than simplification. However, one can now treat the equation as an inhomogeneous, <u>linear</u> equation with variable coefficients, which should make it more tractable for numerical solution. The only non-linearity which remains is found in Equation 4.7. It should be noted that the non-dimensionalization in terms of the size of the coefficients has not been adversely affected. In fact, the dominance of the first two terms has been enhanced since ξ_x, $\xi_{xx} \sim \varepsilon$ and $\varepsilon << 1$. Alternatively, differentiating Equation 4.4 one obtains

$$(2\beta)(1 + \xi_x)^{2\beta-1} g_x(\xi_{tt} - \psi\xi_{xxt}) + (1 + \xi_x)^{2\beta}(g_{tt}$$

$$- \psi\xi_{xxt} - \xi_{xxt}) = g_{xx} \tag{4.8}$$

which really looks as if this "simplification" could have been done without. However, if one multiplies through by $(1 + \xi_x)$, the following equation is obtained

$$(2\beta)(1 + \xi_x)^{2\beta}(\xi_{tt} - \psi\xi_{xxt})g_x + (1 + \xi_x)^{2\beta+1}(g_{tt} - \psi g_{xxt}$$

$$- \xi_{xxt}) = (1 + \xi_x)g_{xx} \qquad \text{or}$$

$$2\beta\xi_{xx}g_x + (1 + \xi_x)^{2\beta+1}(g_{tt} - \psi g_{xxt} - \xi_{xxt}) = (1 + \xi_x)g_{xx} \tag{4.9}$$

which is of the same form as Equation 4.6. This now is the equation which will be utilized to solve the finite amplitude plane wave propagation problem without approximation, save those which may be necessary to obtain a base solution. In this case, this allows a significant extension because this equation does not require the approximation $\alpha/k << 1$, and therefore, the Fubini solution may be used as a base solution for the problem of propagation in highly viscous fluids since it is limited by Mach number only.

The quadrature required to extract the desired solution is essentially dependent on the problem itself. Landahl and Ruppert [13] employed Runge-Kutta integration to solve a boundary-layer problem. In many cases, it is not necessary to resort to such exotic techniques; in this analysis, g varies quite

slowly for small values of ψ; a rapid non-linear change in g with ψ only occurs as ψ becomes very large, that is, as the viscosity becomes high. For most fluids, including very viscous ones, a value of ψ sufficient to produce rapid variation of g with ψ would not be attained at any reasonable acoustic frequency. Consequently, the integration technique employed in this analysis was a very simple straight line integration, which required one iterative step. The initial solution of the equation for g on the ith step was computed based on the ith value of ψ and a solution $\xi_i(x, t)$ which was obtained in the following manner

$$\xi_{i_1}(x, t) = \xi_{i-1}(x, t) + g_{i-1}(x, t) \cdot (\psi_i - \psi_{i-1}) \quad (4.10)$$

With this new solution, $g_i(x, t)$, a modification of $\xi_i(x, t)$ was performed as follows:

$$\xi_{i_2}(x, t) = \xi_{i_1}(x, t) + \frac{1}{2}\big(g_i(x, t) - g_{i-1}(x, t)\big)$$

$$\cdot (\psi_i - \psi_{i-1}) \quad (4.11)$$

The net result being:

$$\xi_i(x, t) = \xi_{i-1}(x, t) + \frac{1}{2}(g_i(x, t) + g_{i-1}(x, t))$$

$$\cdot (\psi_i - \psi_{i-1}) \quad (4.12)$$

which is in essence a trapezoidal integration. Clearly, this scheme would not be useful for g functions which vary rapidly and non-linearly with ψ, that is, if Equation 4.7 exhibits highly non-linear behavior. This scheme could be used in an iterative manner, however, with continuing refinement of the g values at each step. The entire process used to arrive at the extended solution for $\xi(x,t)$ is shown schematically in Figure 2. It may be noted that there is a requirement to obtain variable coefficients at each step, this of course is a situation which is not enviable in a numerical solution in which the coefficients cannot be described analytically, particularly when these variable coefficients involve derivatives of the dependent variable. In this case, all the coefficients involve derivatives. A simple solution to this problem has not been found; however, in this analysis, numerical differentiation of the ξ solution to obtain the coefficients does not seem to have affected the results.

Another not so simple problem in the application of parametric differentiation is the treatment of the boundary conditions. One must be extremely careful in not trying to read too much into the problem. In this analysis, two approaches were considered for determination of boundary values for application in the finite difference scheme, the result being that the most natural and straightforward approach was eventually chosen as the proper one. In analyzing this problem, one can

make some subjective evaluations of the expected behavior of g based on what the $g\Delta\psi$ product represents. Dissipation will tend to slow the non-linear generation of harmonics as the wave propagates, and if dissipation is strong enough, one would expect the waveform to remain sinusoidal, if such is the nature of source excitation, no matter how strong the non-linearity is in absolute terms, that is, for $\Gamma < 1$ the waveform should not steepen appreciably. In any case, near the boundary the effect of $g\Delta\psi$ should be to decrease the amplitude of the signal. $\Delta\psi$ is always positive, therefore to obtain such a reduction in amplitude, one would expect the sign of g to be opposite that of the base solution. Attempting to produce an analytical model which fulfills one's expectations near the boundary may be self defeating. In this analysis it was initially considered desirable to introduce a small perturbation of the boundary condition in terms of ψ, and consequently the boundary condition was rewritten as,

$$\xi(0,\ t) = -\frac{\Xi_o}{(1+\psi)}\cos(\omega t) \tag{4.13}$$

which yielded the intuitively desirable result in terms of sign,

$$\frac{\partial}{\partial\psi}(\xi(0,\ t)) = g(0,\ t) = \frac{\Xi_o}{(1+\psi)^2}\cos(\omega t) \tag{4.14}$$

This model also seemed appropriate since it produced a bound-

ary value of g which decreased slightly as ψ increased, corresponding to an anticipated increase in the viscous effects. If one recalls the nature of the quadrature, the only effect of this model would be to force the solution near the boundary to take on the desired form. This 'forcing' of the solution by injecting one's subjective expectations into the modelling of the boundary conditions was not particularly bad, but neither did it accomplish anything with respect to improving the solution. If one instead blindly applies the $\frac{\partial}{\partial\psi}$ operator to the boundary condition, the result is

$$g(0,\ t) = \frac{\partial}{\partial\psi}(-\Xi_o \cos(\omega t)\ ) = 0 \qquad (4.15)$$

which does absolutely nothing to convince one that the proper sort of quadrature will occur near the boundary, but instead relies on the differential equation to properly predict the g values everywhere. In the final analysis, this was the correct procedure; there were in fact very small differences in the solution values obtained with the forced boundary conditions, and those obtained with the 'natural' condition.

One may wonder why initial conditions are not important in this problem since it is essentially hyperbolic in character, and such problems are generally thought of as initial value problems. The answer is that when one is interested in the steady state solution only, for a problem with a periodi-

cally varying boundary condition, and in this case that is the solution desired for $\xi(x, t)$, one may set the condition that the initial values occurred at some sufficiently large time in the past that all start-up transients have disappeared, and therefore only the boundary conditions determine the behavior of the solution [16]. It is only natural to expect this physical interpretation to be carried over into the parameter space application, and consequently, initial values have been of no concern.

It is valuable to note that the final result for the chosen maximum value of ψ need not be the only output. For a given $\beta\varepsilon$ product, a solution for any Γ consistent with the accuracy obtainable (as $\Gamma \to \infty$, the accuracy required to obtain meaningful results is increased for a given $\beta\varepsilon$ because the differences between the inviscid and viscous solutions go to zero), may be obtained by selecting the appropriate $\psi = \frac{\beta\varepsilon}{\pi\Gamma}$ and simply calling for output at the end of the second iterative step. For a given fluid this would allow one to analyze the losses at different frequencies for a given $\beta\varepsilon$ and, in this case, out to the discontinuity distance. In fact, tabulations of g versus ψ at specific points could be stored so that the solution could be extracted in the future simply by integrating and applying the correction at the appropriate point in the base solution.

It is useful to compare the base solution waveform with

the anticipated extended solution waveform, the comparison in this case being very simple because the extended solution or a very good approximation thereto is well known. As it develops, the incremental differences between the base solution and the extended solution are expected to be in phase with one another, if they are written in the form

$$\Delta\xi(x, t) = \xi_f(x, t) - \xi_o(x, t) \tag{4.16}$$

where $\Delta\xi(x, t)$ is the difference between the calculated solution and the base solution when integrating Equation 4.7 from ψ_o to ψ_f. The observance of $\Delta\xi(x, t)$ varying in the expected manner is a key clue to the success of the application of the method. This behavior was in fact observed in this analysis, and will be discussed in a following section. Once one has developed an equation in parameter space and made a subjective evaluation of what he expects to be the functional form of g and of $\Delta\xi(x, t)$, he is ready to proceed to the numerical technique to be used for solution. Ideally, it would be hoped that the transform to parameter space would lead to an analytic solution of Equations 4.6 or 4.9 and Equation 4.7, or at the very least, Equation 4.6 or 4.9. Such is not the case here, and numerical solution of both was required. Thus, to analyze the results, and prove the worth of the method, a subjective feel for the results was required.

V. FINITE DIFFERENCE EQUATIONS

After the equation of motion has been derived, non-dimensionalized and its analog in parameter space has been obtained through differentiation, it may still be a significant task to obtain the solution of the linear g equation. In this case, Equation 4.6 or 4.9 describes the function g with boundary condition $g(0, t) = 0$. In analyzing the equations, it is noted that they are essentially wave-like in nature, at least close to the boundary, since the g_{xx} and g_{tt} terms dominate. Adopting the approximation $\xi_x \sim \xi_t$ (in non-dimensional form) and differentiating with respect to x, one notes that $\xi_{xx} \approx -u_x$ and since $u_x \to -\infty$ at the shock in the inviscid case, then $\xi_{xx} \to \infty$ at the shock. This wave-like behavior may not then be so readily observable away from the origin since the coefficients involving ξ_{xx} may become very large in the regions in which the shock is forming, or has formed in an extended solution, in which case the g_x term becomes important. The g_{xxt} term remains small since ψ is generally much less than one. The g_x term, of course, is the one which results from the inclusion of non-linear terms, and it is entirely consistent that it becomes important in the regions of the waveform which exhibit the result of non-linear propagation, namely in the region at the head of each compressive portion of the waveform and at the tail of the rarefactive portion. Once a shock has formed, it is conceivable that the g_x term will alter the character of

the equation to the extent that it is no longer wavelike; this is not of immediate concern since this analysis is restricted to the first region of propagation. Therefore, in attempting to solve this equation, it was treated as if it were wavelike throughout. As previously noted, the literature is rich with various methods of solution of the fundamental equation of motion, techniques which, if applied judiciously and with essentially the same approximations as those utilized to solve the equation of motion, could possibly be applied to Equation 4.6 or 4.9 to yield analytic solutions. However, because of the form of the variable coefficients, this analytic solution would at best be extremely difficult and most likely would be impossible; so in this analysis solution, by finite differences, of the parameter space equation was accomplished, and numerical integration of Equation 4.7 was performed. Since the equation is essentially hyperbolic in nature, a finite difference scheme appropriate to a hyperbolic system was the obvious choice.

There are two broad categories of finite difference schemes for partial differential equations, which are termed explicit and implicit. Adopting a standard notation for a nine point difference mesh, Figure 3, an explicit scheme, is one which has only one unknown in the i+1 column or the j+1 row, depending on whether one is "marching" in the i or j direction. An implicit scheme is one for which there are two or

more unknowns in the i+1 column or j+1 row. To solve the explicit scheme for a single value in the future (marching direction) one need know only the values of the dependent variables at all present and past positions on the mesh. The terms future, present and past apply in a spatial as well as temporal sense. Thus, for the explicit scheme, if one has initial conditions valid for all space, or if one has initial conditions for a half space and a boundary condition, a complete solution can theoretically be generated numerically for a properly-posed problem. The implicit scheme requires solution at two or more points based on present and past points. Thus, one generates two unknowns in the first application of the mesh, and an additional unknown with each subsequent application, yielding N+1 unknowns with N equations. Thus, an additional condition is required in order to apply an implicit scheme; generally it is a boundary condition at some point in space. In any case, this requirement essentially restricts the use of implicit schemes in infinite domain problems such as this one.

Having been limited to the use of explicit solution schemes, there exists one additional problem of significant import; that is, stability of the scheme. Unfortunately, it is difficult to consider stability until one has actually developed the finite difference equation, stability of the scheme being a function of the way the scheme itself is developed.

Proceeding to finite difference approximations of the Equation 4.6 or 4.9, one finds a literature which is rich in references to the wave equation for infinitesimal amplitude waves, in a lossless medium, but which has very little in the way of reference to non-linear equations or equations for a viscous medium. Recalling that the parameter space equation is essentially wavelike as long as the coefficient of the g_x term is relatively small, one can use the finite difference approximations to the wave equation as a starting point. Since an essential aim of this study was to demonstrate the utility of the method of parametric differentiation in non-linear wave problems, a second order finite difference scheme was selected; it is clear that greater accuracy could have been obtained with a higher order scheme. The following finite difference approximations were used for each of the derivatives of g in either Equation 4.6 or 4.9:

$$g_{xx} = (g_{i-1,j} - 2g_{i,j} + g_{i+1,j})/(\Delta x)^2 + O((\Delta x)^2) \qquad (5.1)$$

$$g_{tt} = (g_{i,j-1} - 2g_{i,j} + g_{i,j+1})/(\Delta t)^2 + O((\Delta t)^2) \qquad (5.2)$$

$$g_x = (g_{i+1,j} - g_{i-1,j})/2\Delta x + O((\Delta x)^2) \qquad (5.3)$$

$$g_{xxt} = (g_{i-1,j} - 2g_{i,j} + g_{i+1,j} - g_{i-1,j-1} + 2g_{i,j-1} - g_{i+1,j-1})/\Delta x^2 \Delta t + O((\Delta x)^2(\Delta t)) \qquad (5.4)$$

where Δx and Δt denote the step sizes in space and time respectively. It must be stressed that these approximations, particularly Equations 5.3 and 5.4, are not the only ones which could be used. For instance, a forward or backward difference could have been used in place of the central difference of Equation 5.3. This would, however, have changed the order of the approximations. The form of Equation 5.4 is that of a second central difference in space and a backward difference in time, the backward difference being necessary in order to preserve the explicit nature of the scheme. A central difference or forward difference in time for Equation 5.4 would have not only destroyed the explicit nature of the scheme but would also have yielded an unknown to be determined at the point $g_{i+1,j+1}$ whose coefficient, ψ, would have been much smaller than those of the other terms, and could have led to an instability, even if one were using an implicit scheme. This problem could be resolved in an implicit scheme by substituting for Equation 5.1 a weighted time average of second order approximations of g_{xx}.

Representing the variable coefficients of g_{xx}, g_{tt}, g_x, and g_{xxt} by A_1, A_2, A_3, and A_4 respectively, Equation 4.6 or 4.9 becomes, upon substituting the finite difference approximations 5.1-4:

$$-A_4 \cdot g_{i-1,j-1}/(\Delta x)^2\Delta t + (A_1/(\Delta x)^2 + A_3/2\Delta x + A_4/(\Delta x)^2\Delta t)$$

$$\cdot g_{i-1,j} + (-A_2/(\Delta t)^2 + 2A_4/(\Delta x)^2\Delta t)\cdot g_{i,j-1} + (-2A_1/(\Delta x)^2$$

$$+ 2A_2/(\Delta t)^2 - 2A_4/(\Delta x)^2\Delta t)\cdot g_{i,j} - A_2\cdot g_{i,j+1}/(\Delta t)^2$$

$$- A_4\cdot g_{i+1,j-1}/(\Delta x)^2\Delta t + (A_1/(\Delta x)^2 - A_3/2\Delta x + A_4/(\Delta x)^2\Delta t)$$

$$\cdot g_{i+1,j} = F \tag{5.5}$$

where F is the inhomogeneous term. At this point, one may look at Equation 5.5 and note that it has one unknown in the j+1 row if one is marching in space, and two in the i+1 column if marching in time; so that it is well suited to treatment of an initial value problem explicitly, but is implicit if used for a boundary value problem. In order to apply the scheme in an explicit manner to the boundary value problem, one must make an additional approximation. To solve for g(i+1,j), one needs information in the i-1 and i columns, plus information at the point $g_{i+1,j-1}$. Because the third order derivative is present, there is no way to avoid this problem. However, considering the fact that the coefficient of the third order derivative is small compared to the coefficients of the other terms, it would not be unreasonable to presume that if one had an approximate value of $g_{i+1,j-1}$, then a solution of reasonable accuracy could be obtained for $g_{i+1,j}$, which would be based on five pieces of accurate information and a single piece of approximate infor-

mation. Considering Figure 4, which depicts the finite-difference grid with the coefficients of the various points in the solution cube annotated thereon, one notes that the approximation required at $g_{i+1,j-1}$ need be made only once in each spatial step. After solving for $g_{i+1,j}$ at the 'base' of a single spatial column, $g_{i+1,j-1}$ is now known for all subsequent applications of the cube as one marches out in time. The approximation used to arrive at this "corner point" is a Taylor's series expansion in which the derivatives at the point about which the expansion is taken are approximated by backward differences accurate to $O(\Delta x)$. Again greater accuracy is obtainable, but was not required because of the relative smallness of the coefficient. The resulting expression for the "corner point" is,

$$g_{i+1,j-1} = 2.5 \cdot g_{i,j-1} - 2 \cdot g_{i-1,j-1} + .5 \cdot g_{i-2,j-1} \tag{5.6}$$

It must be noted that this approximation is a potentially weak link in the method, due to the fact that near the head of the waveform the x derivatives become large, and a truncated Taylor's series may not be of sufficient accuracy. In any case, it did not appear to significantly affect the nature of the results in this analysis.

In the discussion of explicit finite difference schemes one must inevitably be concerned with the stability of the

scheme as it relates to step size. According to the Courant-Friedrichs-Lewy criterion for stability of finite difference schemes [17], the ratio of the step sizes must be such that one does not attempt to find the solution at a point which is outside the region of influence of the known data, the region of influence being determined by the characteristics emanating from the endpoints of the known data. Unfortunately, this problem is difficult to address here because of the nature of the governing differential equation. Typically, third-order partial differential equations are not treated as such, but are instead reduced to some more tractable form; in this case the approach using the Burgers equation was such a reduction. Consequently, little appears in the literature with reference to stability for third-order equations. An unsuccessful investigation was conducted to attempt to arrive at the exact characteristics by reduction to canonical form. However, this equation is wave-like, and one could reasonably expect the characteristics to be similar to those of the wave equation, and that the stability criterion would be similar in form. Working with this foreknowledge and applying a modified Von Neumann [17] stability test, an exact stability criterion will be established. The VonNeumann test essentially consists of examining possible exponential solutions to a finite difference scheme to determine when they may grow without bound given finite initial or boundary conditions. With this know-

ledge one may adjust the stepsize so that the scheme will be stable. Unfortunately, the VonNeumann test is applicable to finite difference equations with constant coefficients; in this case the coefficients are variable, and the only recourse is to treat the coefficients as if they were constant, derive the stability criterion, and then discuss the implications of their variability with respect to how it affects the stability criterion.

The exponential solutions to the scheme which may grow without bound will have the form,

$$g_{m,n} = e^{im\theta}e^{in\lambda} \tag{5.7}$$

where the subscript notation m,n corresponds to the notation i,j used previously. λ is any real number and θ is an arbitrary complex number. Thus, boundary data may be written,

$$g_{o,n} = e^{in\lambda} \tag{5.8}$$

and the right hand side may be considered as a typical term in a Fourier expansion of the boundary data. Substituting Equation 5.7 into 5.5, ignoring the inhomogeneous term since it will not affect stability, and dividing through by $e^{im\theta}e^{in\lambda}$, one obtains

$$-A_4e^{-i\theta}e^{-i\lambda}/(\Delta x)^2\Delta t + (A_1/(\Delta x)^2 + A_3/2\Delta x + A_4/(\Delta x)^2\Delta t)e^{-i\theta}$$

$$+ \left(-A_2/(\Delta t)^2 + 2A_4/(\Delta x)^2\Delta t\right)e^{-i\lambda} + \left(-2A_1/(\Delta x)^2 + 2A_2/(\Delta t)^2\right.$$

$$\left. - 2A_4/(\Delta x)^2\Delta t\right) - A_2 e^{i\lambda}/(\Delta t)^2 - A_4 e^{i\theta}e^{-i\lambda}/(\Delta x)^2\Delta t$$

$$+ \left(A_1/(\Delta x)^2 - A_3/2\Delta x + A_4/(\Delta x)^2\Delta t\right)e^{i\theta} = 0 \tag{5.9}$$

Rearranging terms such that they are grouped by coefficients, the following results,

$$A_1\left(\frac{e^{i\theta}+e^{-i\theta}-2}{(\Delta x)^2}\right) - A_2\left(\frac{e^{i\lambda}+e^{-i\lambda}-2}{(\Delta t)^2}\right) - A_3\left(\frac{e^{i\theta}-e^{-i\theta}}{2\Delta x}\right)$$

$$+ A_4\left(\frac{e^{i\theta}+e^{-i\theta}-2}{(\Delta x)^2}\right)\left(\frac{1-e^{-i\lambda}}{\Delta t}\right) = 0 \tag{5.10}$$

Recognizing the various terms as trigonometric forms, one obtains

$$- \left(4A_1\sin^2\frac{\theta}{2}\right)/(\Delta x)^2 + \left(4A_2\sin^2\frac{\lambda}{2}\right)/(\Delta t)^2 - iA_3\sin\theta/\Delta x$$

$$- A_4\left(4\sin^2\frac{\theta}{2}\right)\left(1-e^{-i\lambda}\right)/(\Delta x)^2\Delta t = 0 \tag{5.11}$$

Now solving for the imaginary portion,

$$-A_3\sin\theta/\Delta x - 4A_4\sin^2\frac{\theta}{2}\,\sin\lambda/(\Delta x)^2\Delta t = 0 \tag{5.12}$$

which identity indicates that the product,

$$\Delta x \Delta t = -4A_4 \sin^2\frac{\theta}{2} \sin\lambda / A_3 \sin\theta \tag{5.13}$$

should hold, but this is clearly absurd since it indicates that as $\Delta x \to 0$, Δt may become arbitrarily large as well as negative. Reason and foreknowledge of the stability criterion for the wave equation dictate that this result has no meaning. Returning to Equation 5.11 and equating the real parts, one obtains

$$(-A_1 \sin^2\frac{\theta}{2})/(\Delta x)^2 + (A_2 \sin^2\frac{\lambda}{2})/\Delta t^2 - (A_4 \sin^2\frac{\theta}{2})(1-\cos\lambda)/(\Delta x)^2 \Delta t$$

$$= 0 \tag{5.14}$$

or

$$\sin^2\frac{\theta}{2} = \frac{A_2 (\Delta x)^2 \sin^2\frac{\lambda}{2}}{A_1 (\Delta t)^2 + A_4 \Delta t (1-\cos\lambda)} \tag{5.15}$$

Now one argues that to prevent the assumed exponential solution from increasing without bound, θ must have no negative imaginary part. However, because of the quadratic form, if θ has imaginary roots, they will appear in conjugate pairs; therefore θ must be real. Since λ was assumed arbitrary, this requires the following restriction,

$$0 \leq \frac{A_2 (\Delta x)^2}{A_1 (\Delta t)^2 + A_4 \Delta t (1-\cos\lambda)} \leq 1 \tag{5.16}$$

Now, noting that the maximum value of the expression above is obtained when $1-\cos\lambda = 0$, it is clear that a sufficient expression for the above is,

$$0 \leq \frac{A_2(\Delta x)^2}{A_1(\Delta t)^2} \leq 1 \tag{5.17}$$

which is of precisely the same form as the stability criterion for the wave equation, in which case $A_1 = A_2 = 1$. In this case A_1 and A_2 are variable with $A_1 = 1 + \xi_x$ and $A_2 = (1+\xi_x)^{2\beta+1}$ for Equation 4.9. Since ξ_x is approximately equal to the local acoustic Mach number, which is taken to have a value of $O(10^{-2})$ in this analysis, the criterion is only slightly different from that of the wave equation. However, it is important to realize that the ratio of step sizes cannot be unity as one may have anticipated. It is also comforting to note that A_3, the coefficient of the g_x term, did not enter into the criterion in any way, lower order terms generally having little effect on stability. This statement, however, deserves further comment. Recalling the discussion of the size of the respective terms in the parameter space equation, the g_x term has a coefficient, ξ_{xx}, which will become very large in the region of a shock. It is inconceivable that this term will not affect the stability of the scheme in such a case. The point is that when ξ_{xx} becomes very large the entire nature of the equation does change with the g_x term being the critical

term. In this instance the stability analysis is no longer valid, and one would be forced to rely on some other stability test. This problem is not great here because the analysis is restricted to the first zone of propagation. Also, the discretization of the displacement data limits the maximum value that ξ_{xx} can attain in the numerical scheme. This is, as yet, an unresolved numerical problem; to allow ξ_{xx} to attain its true value in a nearly inviscid fluid, an exceedingly small step size in space would be required. However, if one considers the actual size ξ_{xx} may attain in a typical propagating wave, this problem may be put in a proper light. Blackstock [5] has defined a shock thickness for the viscous problem,

$$T = 2\Delta\cosh^{-1}\sqrt{\pi/\Delta} \tag{5.18}$$

where T is the thickness, defined as the spatial distance from trough to peak of the waveform, and $\Delta = 2(1 + \sigma)/\pi\Gamma$. As an example, consider the case when $\Gamma = 100$ and $\sigma = 1$; then $T = \frac{8}{100\pi}\cosh^{-1}5\pi \approx .08$. In this case, ξ_{xx} changes from a maximum negative value to a maximum positive value over a change in x of .08. Since $\xi_{xx} \approx -u_x$, the change in u over this distance is approximately twice the Mach number, which as noted before is taken to be of $O(10^{-2})$. Then an approximate value of ξ_{xx} may be taken as $\Delta u/\Delta x \approx .25$; therefore, while the g_x term does become important, it does not dominate. For larger values of

Γ, it will become increasingly important, will in fact dominate the equation, and will require smaller step size near a shock to correctly estimate ξ_{xx}.

Another consideration in the finite difference scheme is the handling of boundary values. Treated as an initial-value problem, there would be no problem in starting the solution, since ξ and ξ_t would be everywhere prescribed for $t = 0$. In this case, only one boundary condition is available for $x = 0$, the other condition being that $\xi \to 0$ as $x \to \infty$, which is of no import here except that it required the ruling out of solutions which grow in space. Because two "columns" of information are needed at the boundary to start the solution for g, another condition is required. Recalling the discussion in the previous chapter concerning the modelling of boundary conditions, this problem of requiring an extra condition could lead one to expend great effort in attempting to develop a consistent and meaningful extra condition, when in fact if one simply applies the method of parametric differentiation as it is defined, the second condition will become apparent. The point is that the boundary is a real thing, not some abstraction in parameter space, and consequently the source excitation has no dependence on ψ whatsoever; therefore one can in fact, without worrying about what ξ_x is at the boundary (which, incidentally, would overspecify the problem), write another boundary condition,

$$g_x(0, t) = 0 \tag{5.19}$$

This completes the information needed to march out the solution of the problem. It is interesting to note that because g and g_x are zero at the boundary, the only non-zero term in the finite difference cube on its first application is the inhomogeneous term. Without this term, there would be no solution because all terms in the finite difference cube would be zero. Again, this is consistent with intuitive thoughts about the form of the solution for g; it should in fact be very small near the boundary so that when the quadrature is performed to calculate the particle displacement, the net change in the value of ξ from the inviscid to the viscous solution is small.

VI. RESULTS

The practical realization of the preceding development is a computer program which solves the non-linear plane wave propagation problem in a viscous fluid and provides output which allows one to readily compare the results obtained with this method, and those obtained by Keck and Beyer [4]. Their solution was chosen as the point for comparison because it was simple and easy to calculate, and because this analysis had to be conducted on a shoestring budget. Unfortunately, this did not allow good comparisons for the higher ranges of Γ because the Keck/Beyer solution is not sufficiently accurate [6], lacking harmonic content beyond the sixth harmonic. One can argue though that the results for large Γ appear to be consistent with one's expectations, and that the method appears to generate a plausible progression of results as the value of Γ is decreased (corresponding to an increase in ψ). An annotated version of this computer program appears in Appendix A. It is worthwhile to note at this point that because of the hyperbolic nature of the problem, the domain of dependence for a given point becomes larger and larger the further away from the boundary that the point is located. The result is that if one desires a solution value N discrete steps away from the boundary, $N^2 + 2N + 1$ points are required in the solution grid. This is a fundamental limitation of this method as it is presently formulated, and may be difficult to improve upon.

In this analysis the Mach number was taken to be rather large, $\varepsilon = O(.01)$, in order to minimize N while still being able to generate useful results. B/A was taken as 6, which is the approximate value of that quantity for water; thus $\beta = 4$. For a given $\beta\varepsilon$ product, the discontinuity distance, $\overline{X}$, will occur at a point which is $1/2\pi\beta\varepsilon$ wavelengths from the boundary, regardless of frequency. Thus, for $\beta\varepsilon = .04$, $\overline{X} \approx 4\lambda$. In water, a Mach number of .01 corresponds to an acoustic pressure given by the expression $p = \rho_o C_o^2 \varepsilon$ or $p \approx 225$ atmospheres, or in decibel units, SPL = 267 dB re 1 uPa, which is quite a strong signal. The values of ψ have been varied from zero to .01 which would reflect a change from inviscid to highly viscous propagation. Recalling the form $\psi = \nu b/C_o\lambda$, it is clear that by fixing ψ, the frequency for which the solution is valid is also fixed for a given fluid. To put these results in a proper perspective, a value of $\psi = .01$ in water would correspond to a frequency of approximately five gigahertz. However, the results for smaller values of ψ represent more reasonable frequencies. Another interpretation of the results for $\psi = .01$ is that a solution has been obtained for the fluid with an extremely high viscosity coefficient such as glycerin, a solution which has not previously been obtained analytically, because of the restriction $\alpha/K \ll 1$, which is necessary to reduce the equations to analytically tractable forms. One might conclude from these results that there is nothing significantly

different in the form of the solution for such a liquid. In any case, the results which follow will indicate the potential of this method for application over a great range of values of the non-linear and viscous parameters.

Figures 5a, b, c, and d are plots of the particle displacement calculated by parametric differentiation, the particle velocity calculated from this solution, the difference between the inviscid (Fubini) and the viscous (Keck-Beyer) displacement solutions, and the difference between the inviscid and viscous (parametric differentiation - P.D.) displacement solutions, respectively, for $\varepsilon = .01$, and $\psi = .01$ ($\Gamma = 1.27$). The two plots of the calculated difference are the basic results used for evaluation of the parametric differentiation solution. The differences are calculated according to Equation 4.16, and one would expect the calculated differences to be in phase with the displacement solutions; since viscosity should reduce the absolute amplitudes, the difference should be maximum when the Fubini solution has a maximum and minimum when the Fubini solution has a minimum. The zeroes will not necessarily coincide since the effect of the non-linearity is to lengthen the compressive portion of the waveform while reducing its magnitude and to shorten the rarefactive portion while increasing its absolute magnitude; viscosity will tend to keep the waveform sinusoidal, thus maintaining the length of the two portions of the waveform. Thus one could

hardly expect the zeroes to coincide.

Since $\Gamma = 1.27$, one can infer that the Keck-Beyer solution should be accurate, and also that some harmonic content should be evident in the particle velocity solution, assuming $\Gamma = 1$ is an accurate representation of the incipience of important non-linear effects. In analyzing Figures 5c and 5d, one is immediately struck by the fact that although the calculated differences seem to be of the same form, there seems to be some sort of ramp type behavior entering into the P.D. displacement solution. If one were to try to place a physical interpretation on this result, it would be that the ability of the medium to restore itself elastically has been exceeded due to the large displacement amplitudes and that therefore a permanent displacement has been induced, just as if a spring had been stretched beyond its elastic limit inducing a permanent strain. Due to the fluid nature of the medium and because no analytic solution predicts such behavior, such a physical interpretation may not have much meaning. An attempt will be made to offer a possible explanation for this result in a later paragraph, but this remains an unsolved problem in this work. The average numerical loss in the positive portion of the waveform is clearly greater than that of the negative portion, rather than being equal as one would expect.

The velocity solution, on the other hand, seems to be quite normal. One can observe that though non-linear effects

are producing some steepening of the profile, viscous dissipation does in fact reduce the amplitudes properly; the amplitude of the last peak is 3.1 dB down vice 3.3 dB down by the Keck-Beyer solution, a difference of about 6%. The last point in the velocity profile is numerically inaccurate because of the lack of sufficient points at the tip of the solution mesh. The small "bump" in the velocity profile is not easily explained, but probably arises from numerical inaccuracies. It seems remarkable that with little direct effort aimed at reducing computational inaccuracies in the numerical scheme, the difference in the solutions is but 6%, remembering all the while that this difference may not represent an error at all, but rather a more accurate solution produced by a method which does not resort to common approximations. One may wonder at this point why an equation in particle displacement vice one in particle velocity was treated, the answer being that the equation in particle velocity would still have required one to know the displacement solution in order to arrive at the variable coefficients which would have appeared in the parameter space equation, and thus would probably have been no essential simplification. Another aspect of the P.D. solution which is worthy of note is the steepness of the velocity profile. The Keck-Beyer solution for this value of Γ looks essentially like a decaying sinusoid, while the P.D. solution appears to indicate that the non-linearity remains fairly important, result-

ing in some steepening of the profile. This suggests that perhaps $\Gamma = 1$ may not be a sufficiently strong indicator of the incipience of non-linear effects.

Results analogous to those of Figure 5 have been obtained for different values of the problem parameters, but with the number of points utilized in the calculation maintained at N = .01. Figures 6 and 7 are the plotted results for $\varepsilon = .01$ and $\psi = .001$ ($\Gamma = 12.7$), and for $\varepsilon = .01$ and $\psi = .0000463$ ($\Gamma = 275$) respectively. Figures 8, 9, and 10 are the results for $\varepsilon = .02$ and $\psi = .01$ ($\Gamma = 2.55$), $\psi = .001$ ($\Gamma = 25.5$), and $\psi = .0000463$ ($\Gamma = 550$) respectively. In addition, calculations have been conducted for $\varepsilon = .0075$ and $\varepsilon = .005$, but are not shown here. These results are not much different in form from those discussed previously, but they uncover some problems with the solution technique as well as help demonstrate that the method is in fact a potentially useful one for investigation of non-linear wave propagation problems, since it successfully solves the equation of motion in a predictable and consistent manner for various values of Γ. The range of the $\beta\varepsilon$ product had to be restricted as it was because of the need to minimize computer costs and thus the number of points at which the difference equation was to be solved.

Shifting attention to Figure 6, first recall that with $\Gamma = 12.7$ the Keck-Beyer solution is probably only marginally accurate. The displacement solution again exhibits the ramp type

behavior, although it is only readily clear when analyzing Figure 6d, which shows a greater average loss in the positive portion of the waveform than in the negative portion. Figure 6c shows that the Keck-Beyer solution is in fact no longer accurate near the discontinuity distance. The velocity waveform appears to be consistent with expectations with only slight attenuation because of the relative smallness of the attenuation coefficient over the spatial length of the calculations. However, small signal attenuation, i.e., that attenuation which would occur if no harmonics were non-linearly produced, would predict a local Mach number of .0092 at the discontinuity while in this case a value of .0096 was obtained. This result is not consistent; one would expect greater attenuation in the non-linear case because the generation of harmonics will reduce the magnitude of the primary in addition to small signal attentuation, while the harmonics themselves will be attenuated even more rapidly because of greater attenuation coefficients. Unfortunately, in this case there is no easy explanation of the result; however, it did prompt some thoughts about what the Fubini solution looks like and what the changing shape of the waveform should imply with respect to the instantaneous values of particle velocity near the discontinuity distance.

From the point of view of energy conservation, one would expect that in a fluid without viscous dissipation, the time

or space average energy density in a plane wave train propagating isentropically will remain constant. The implication of this is that as the finite amplitude waveform steepens and eventually assumes a sawtooth shape, the peak values of the particle velocity must increase in order to maintain constant average energy density in the wave train. For a sawtooth wave to have the same energy density as a sinusoid of unit amplitude and identical period, its amplitude must be approximately 1.22. Realizing that a fully formed sawtooth develops at a distance from the boundary of about $3.5\overline{X}$ [3], it would still be expected that the peak value of the particle velocity at the discontinuity distance would be greater than that at the boundary. The base solution (Fubini) calculated in this study is accurate to five places in displacement and three in velocity, and the peak particle velocity at the discontinuity distance is identical to that at the boundary, implying that there in fact has been an energy loss in an inviscid solution, and it is certainly not clear where this energy has gone. In any case, for a Γ of 12.7, non-linear effects are definitely important, and it is entirely conceivable that the P.D. solution has somehow recognized an error in the base solution and has yielded a quadrature value which correctly accounts for the steepening of the profile and thus predicts a particle velocity which is greater than that which would be obtained with a sinusoid and small signal attenuation. This explanation is of course sup-

position since there are too few points calculated to obtain a good space average energy density to verify an energy loss which is greater than that predicted by small signal attenuation. It indicates an unsolved problem and at the same time suggests that parametric differentiation predicts the proper solution to the problem. Blackstock [9] suggests the mechanism for dissipation in an inviscid fluid after a shock has formed; here, there is no reason for such dissipation since there are no shocks in the first zone of non-linear propagation.

Figure 7 shows much the same results as those of Figures 5 and 6. In this case, $\Gamma = 275$, and the Keck-Beyer solution is clearly not converging to the proper result as evidenced by Figure 7c. Again the ramp behavior is clear in Figure 7d, though the magnitudes of the difference values are understandably smaller. The peak value of the calculated Mach number at the discontinuity distance is now .0106 which is in fact greater than the peak at the boundary. The differences between the inviscid and viscous displacement solutions are now so small that on the scale of these plots, the viscous solution appears identical to the inviscid solution. This is not remarkable, but it is noteworthy that the plotted differences, though small, produce a smooth curve rather than one that jumps from point to point. Figures 8, 9, and 10, which represent the calculations for a source Mach number of .02, show results

which are similar to those shown in Figures 5, 6, and 7, and do not need to be analyzed with as much detail. However, in Figure 8, with $\Gamma = 2.55$, the peak Mach value near the discontinuity distance is .0147 vice .0137 according to the Keck-Beyer solution for a difference of 7%, which is consistent with the 6% difference obtained in the case $\varepsilon = .01$, $\Gamma = 1.27$. It is also consistent with the idea that peak Mach numbers should increase in the non-linear case since small signal attenuation would predict a Mach number of .014 at the same point for a purely sinusoidal wave. The ramp behavior is evident in each of the Figures 8d, 9d, and 10d. Finally, these results also provide a counter example for the idea that the peak Mach number should increase near the discontinuity distance in the case of strong non-linearity, i.e. when Γ is greater than 10. The particle velocity plots of Figures 9b and 10b indicate peak Mach values which remain less than the peak at the boundary; in Figure 9b the value is .0193 and in Figure 10b it is .01998. Small signal attenuation would predict a value of .0192 for the result of Figure 9b. Energy considerations dictate an increase in the peak Mach values and in this case it does not occur. However, the spatial step size used to arrive at the plots of Figures 6 and 7 was .04, while in the calculations used to obtain the plots of Figures 9 and 10, the spatial step size was .02. Recalling the finite difference approximations utilized, this difference in step size may be the

explanation for the difference in results. The truncation error for the second-order difference approximations is of $O\left((\Delta x)^2\right)$. Therefore, it is certainly conceivable that the step size of .04 is not accurate enough to yield good results for particle velocity. At the same time the considerations of energy cannot be ignored. It is likely that the increase in the peak Mach number in the results of Figures 6 and 7 is a result of numerical inaccuracy, but this is certainly not entirely clear. The other calculations conducted with Mach numbers of .0075 and .005 produced results which were consistent with the other results obtained, but they are not presented here because the spatial step size was sufficiently large that the results are somewhat questionable because of truncation error.

There remains a question with respect to the curious ramp behavior in the particle displacement solutions. In examining the plots of the difference between the inviscid and viscous solutions, this behavior is always clear, but would not necessarily be clear at all in analyzing the particle displacement results, particularly as ψ becomes smaller and smaller. ψ is a direct measure of the viscosity of the assumed fluid rather than a relative measure of viscosity such as Γ. Recalling that a ψ of .01 would imply a highly viscous liquid or a very high frequency wave and that the approximation upon which previous authors' results are based is $\alpha/K << 1$, which implies weak viscosity, the reason for the ramp may become clear. Previous

analyses, in making this approximation, may have neglected a rather interesting physical interpretation of the non-linear loss mechanism. For small ψ, the values of the calculated differences are so small compared to the values of particle displacement that this effect would not be noted when solving for displacement or velocity without consideration of the differences between the viscous and inviscid solutions. In fact, in making the assumption $\alpha/K << 1$, the values of ψ are apparently restricted to relatively small values; $\alpha/K = \frac{\nu b \omega}{2C_o^2} = \frac{\pi \nu b}{C_o \lambda}$; and $\psi = \frac{\nu b}{C_o \lambda}$, therefore it is clear that previous analyses have not considered values of ψ on the order of those considered here. For large ψ and thus highly viscous propagation, previous results are not valid, and therefore if the ramp behavior has a physical explanation, it could not have been noted before.

In the parameter space Equations 4.6 or 4.9, there are four terms which involve the dependent variable, g_{xx}, g_{tt}, g_{xxt}, and g_x, which result from similar terms in the displacement space Equation 3.9. g_{xx} is a term which relates to the elastic properties of the medium, g_{tt} to the inertial properties, and g_{xxt} to the viscous properties. The g_x term implies that there exists some modification of the elastic properties of the medium. Recalling the discussion in the previous section concerning the effect of this term in the portions of the waveform where the velocity gradient u_x assumes large values,

the source of the ramp behavior may become evident. When ξ_{xx} is large, the g_x term will dominate the equation. In the inviscid solution, ξ_{xx} becomes infinite at the first shock. Since the inviscid solution is in fact the starting solution and the quadrature from parameter space to displacement space is accomplished with an integration with respect to ψ, it is obviously not unreasonable to expect ξ_{xx} to assume large values when ψ is small; also, one would expect ξ_{xx} to assume numerical values which are more accurate for the current ψ step of the calculation if the spatial step size was small in the regions of large ξ_{xx}. In any case, if the g_x term dominates the equation in a specific region, the g solution in that region will vary linearly with x, hence the ξ solution will vary in a similar manner, and hence the ramp.

In the absence of viscosity, it is clear that the fluid particles oscillate about an equilibrium position; with viscosity, realizing that the g_x term is one which, in essence, alters the elastic constant of the system, it is postulated that the following phenomenom occurs. Because of the large velocity amplitudes, the phase velocity increases in the compressive portion of the wave and decreases in the rarefactive portion; this is well known. These large particle velocities produce correspondingly large particle displacements, but now the fluid particles, instead of being purely elastic, do not have all of their kinetic energy at zero displacement converted

to potential energy at full displacement. Some is lost to small signal attenuation of both fundamental and harmonics, and the rest produces an irreversible displacement of the fluid particles; that is, work is expended on the fluid system to produce a displacement of the equilibrium positions of the fluid particles. Unfortunately, this effect can only be postulated here because a precise energy balance has not been performed. In order to calculate the specific energy lost in producing a given displacement at a given point, the net acceleration on the fluid particles which produced this displacement would be required, and this calculation has not yet been performed.

The reason for this apparent displacement of the equilibrium positions of the fluid particles may be explained in a more rhetorical manner as an interaction between non-linear and viscous effects. The non-linear effect is to alter the elastic constant thus changing the speed of propagation of the wave, resulting in the steepening of the waveform. Now, because the oncoming compressive portion of a given cycle of the wave is travelling faster than the preceding rarefactive portion and because viscosity is acting to slow the velocity of the fluid particles in the rarefactive portion even more, the particles, after passage of the given cycle, do not return to their former equilibrium position, but are instead minutely displaced. The oncoming compressive portion of the wave is

'filling' the preceding rarefactive portion and consequently the fluid particles in the rarefactive portion do not have the opportunity to recover their initial equilibrium positions. Such an effect would obviously tend to create a vacuum behind the wave train, but this can be justified by requiring that fluid flow in from outside the region excited by the source.

Finally, there are some matters concerning the numerical method which require attention. The stability criterion derived in the preceding chapter was arrived at in a rather unconventional manner. Some numerical experimentation was performed to determine just how sensitive to the ratio of step sizes the solution was. The exact stability criterion was

$$\Delta x \leq \Delta t/(1 + \frac{\partial \xi}{\partial x})^{2\beta} \tag{6.1}$$

which implies for $\partial\xi/\partial x$ positive, $\varepsilon = .01$, and $\beta = 4$ that $\Delta x \leq \Delta t/1.08$ near the first shock. This should be an absolute limit on the spatial step size since $\partial\xi/\partial x$ will have its maximum value at this point. The equations were solved for $\Delta x=.04$ and $\Delta t=.04$, and the numerical solution became highly unstable beyond the second cycle of the waveform. Then Δt was set equal to .045 which would be just slightly greater than the Δt required by the stability criterion for $\Delta x=.04$, and the solution was stable throughout. All subsequent calculations were conducted with a convenient Δt selected so that it was large enough

for the given Δx to satisfy the stability criterion.

Another matter of concern in the calculations is the effect of the step sizes, in x and t and in ψ as well, on the accuracy of the solution. In this analysis, the base solution was calculated accurate to five significant figures, obviously limiting the accuracy of the calculated solution to the same extent. The base solution required 51 harmonics at the discontinuity distance for this accuracy. The spatial step sizes used for the various solutions were Δx = .08, .06, .04, and .02. Since the truncation errors in the second order differences are of $O((\Delta x)^2)$, it would be reasonable to assume that the calculated values of g would be accurate to two or three places depending on the step size used. However, to return to displacement space, the values of g are multiplied by a difference between two values of ψ, and hence for a maximum ψ of .01, if g is accurate to three significant figures, the differences calculated in the quadrature will leave the displacement solution accurate to five significant figures, neglecting errors introduced by the discretization of the solutions with respect to ψ. This then would be consistent with a base solution calculated to four significant figures, for Δx=.02 and marginally consistent for Δx=.04. This is the reason why the results for ε = .005 and ε = .0075 were not presented; their accuracy could only be considered marginal at best for the maximum ψ value. These were the solutions which employed step sizes of .06 and

.08. Roundoff error was not considered to be of significant concern in this analysis with machine accuracy to seven significant figures. The solutions also did not seem to be sensitive to the step size in time as long as the time step was large enough to satisfy the stability criterion at all points. Finally, the step size in ψ seemed to matter very little in the form of the results if not in the exact numerical results. It is obvious that in integrating across orders of magnitude, the integration must be handled carefully, but experience demonstrated that values of g at a given point changed little as ψ was varied, and thus the error introduced by utilizing a relatively large step size in ψ should be small. Consequently, the following scheme was adopted:

1) the first ψ value was the limiting value ψ_o
2) the second ψ value was that which when used in the quadrature yielded difference values which were just within machine roundoff accuracy
3) each subsequent step in the order of magnitude of ψ was broken into an algebraically increasing number of steps on a logarithmic basis.

For example, with $\psi_o = 0$, $\psi_1 = 10^{-6}$, there would be two steps between $\psi = 10^{-6}$ and $\psi = 10^{-5}$, with $\psi_2 = .316 \times 10^{-5}$ and $\psi_3 = 10^{-5}$, three steps between $\psi = 10^{-5}$ and $\psi = 10^{-4}$, $\psi_4 = .214 \times 10^{-4}$, $\psi_5 = .463 \times 10^{-4}$, and $\psi_6 = 10^{-4}$, etc. There is no

clear method to determine the proper step size in ψ, but this one produced consistent results. In general, it would be inappropriate if g varied in a highly non-linear manner with ψ.

The final comments concerning the numerical results are made with respect to the formulation of the problem. Equations 4.6 and 4.9 were the parameter space equations developed from the equation treated by Keck and Beyer and from the parent equation without approximation, respectively. Early calculations were performed with Equation 4.6 and later ones with Equation 4.9. For $\varepsilon = .01$ and $\psi = .01$, the maximum percentage difference between the calculated solutions by Equations 4.6 and 4.9 was approximately 4%, demonstrating that the equation solved in previous work does in fact produce very little error for Mach numbers up to .01. Finally, the two forms of the boundary values for g appeared to make little difference in the final result; producing a maximum percentage difference in the calculated values of particle displacement of approximately 1% between the case utilizing the 'natural' boundary conditions for g and the case in which artificial dependence on ψ was introduced at the boundary.

VII. CONCLUSIONS AND RECOMMENDATIONS

A theoretical study of non-linear plane acoustic wave propagation in a viscous fluid has been conducted utilizing a little known mathematical technique, parametric differentiation, in order to demonstrate its utility when applied to acoustic propagation problems, and to obtain whatever physical insight into the problem the results may present. It has been demonstrated that the method can in fact yield results which are consistent with other analytic solutions to the problem for certain arbitrarily selected values of the viscous and non-linear parameters. No restriction was required in the selection of these values by the mathematical nature of the method, and there is no reason to believe that the method could not be applied for any values of the parameters. Restrictions which may exist in the application of the method are computational ones, specifically: 1) the results can only be as accurate as the solution to the problem for $\psi = \psi_o$; 2) accuracy may be further limited by the finite difference approximations utilized; 3) accuracy may be limited by the ability of the computer to solve the finite difference scheme for a large number of points with a given time and memory budget; for instance, double precision will require twice the computer memory; and 4) the hyperbolic nature of the problem requires calculations to be conducted throughout the domain of dependence of a given point to arrive at a solution at that point. These restric-

tions may render the method unusable for calculations involving large propagation distances, but they do not detract from its ability to provide physical insight. Further research may provide a reasonable technique for application of the method when large distances are involved. The solution of the problem by parametric differentiation has allowed the elimination of the restriction $\alpha/k << 1$ heretofore assumed in typical propagation studies. Elimination of this restriction has produced results which are valid for highly viscous fluids as well as demonstrated that the results for such fluids are essentially similar to those for less viscous fluids. It has allowed the formulation of a postulated behavior for all viscous fluids, which may explain physically why there are non-linearly induced energy losses above and beyond those associated with viscous attenuation. Specifically the solutions produced by this method predict that some of the energy contained in a finite amplitude wave in a viscous medium is expended in producing a displacement of the equilibrium positions of the fluid particles. These results have also been obtained cheaply; for 101 spatial points, the computer time required to produce a result for single values of ψ and the $\beta\varepsilon$ product is approximately .15 minutes with 400 kilobytes of computer memory.

This is not to suggest that this work proves beyond a doubt that the method is a useful one. Consideration of propagation once a shock has actually formed has not been attempted.

If the shocks were in fact discontinuities as predicted by the inviscid theory, the technique probably would not be useful beyond the first zone of propagation. However, since the waveform is in fact continuous through a shock of finite thickness, it is reasonable to expect that with denser sampling of the waveform through the shocks, good numerical results could be obtained. The next step in a continuing study of the solution of the problem through parametric differentiation would be to select Blackstock's weak shock solution [3], which connects the Fubini solution [1] with the Fay solution [2] in its inviscid limit to yield a solution valid to the end of the second zone of propagation, as the base solution for $\psi_o = 0$, and to attempt to extend the solution into the region of periodic shocks. With this result a precise energy balance calculation would demonstrate whether or not energy is lost in producing a permanent displacement of the fluid particles.

More fundamentally, it would be useful if a mathematical analysis of the technique were to be conducted in order to determine why it apparently correctly predicts solutions for many different classes of problems, and what if any are the theoretical limitations to its application. Beyond this, there are more interesting problems in non-linear wave propagation to which the method could be applied. The first, and one which can be readily adapted to the computer program already written, would be an investigation of the solution of the Burgers equa-

tion as Γ is varied. Having noted previously that this solution is difficult to calculate as Γ becomes much greater than $x/\overline{X}$ or for $\Gamma > 50$ because of slow convergence, it is suggested that the analytic solution be calculated for $\Gamma = 1$. Then with the Burgers equation in the form solved by Blackstock [5], with $V = u/U_o$,

$$V_\sigma - VV_y = \Gamma^{-1}V_{yy}, \quad V(o, y) = \sin\omega y \tag{7.1}$$

differentiate with respect to Γ to obtain,

$$g_\sigma - Vg_y - V_y g = \Gamma^{-1}g_{yy} - \Gamma^{-2}V_{yy}, \quad g(o, y) = 0 \tag{7.2}$$

where σ is a spacelike, and y a timelike variable. This equation would be perfectly suited to solution on the mesh used in this analysis because there would be no value at the $i+1, j-1$ point, and the scheme would be completely explicit. The solution for $\Gamma = 1$ could then be extended to arbitrary values of Γ without significant computational problems. Because of the space scale of Equation 7.1, this technique will allow easy calculation of the solution in both the first and second zones of propagation.

Finally, there are the two problems of finite amplitude wave propagation which are yet to be solved analytically, those of propagation of cylindrical and spherical waves in viscous

fluids; parametric differentiation may provide an invaluable tool in solving these problems. The fundamental equations of motion do not lend themselves to analytic solution, there being no simple transform such as that used by Blackstock in the plane wave problem to reduce the equation of motion to a Burgers equation [18]. Indeed, the inviscid forms of the equation have no exact analytic solutions which have been published. Blackstock [9] has obtained approximate solutions to the cylindrical and spherical wave problems for an inviscid fluid, one-dimensional wave propagation, and $r >> 1$, which are very similar in form to the Fubini solution; these results were obtained through manipulations of the equation of motion which yielded an inviscid Burgers equation. Such solutions could be used as one base solution for a solution to the viscous problem by parametric differentiation, realizing that they are only approximate. Other authors [12][19] have produced particular solutions for spherical and cylindrical wave propagation, but with an accurate base solution, parametric differentiation could be used to generate many solutions, for a fixed $\beta\varepsilon$ product, with relative economy. Consequently, the apparent course of study to pursue for non-linear cylindrical and spherical wave propagation is the development of improved base solutions for $\psi_o = 0$ and the application of parametric differentiation to these solutions, producing a large number of solutions which may then allow for some generalization of the results. Such a

research program could lead to a significant gain in the understanding of the physical mechanisms which are collectively described as non-linear attenuation in viscous fluids.

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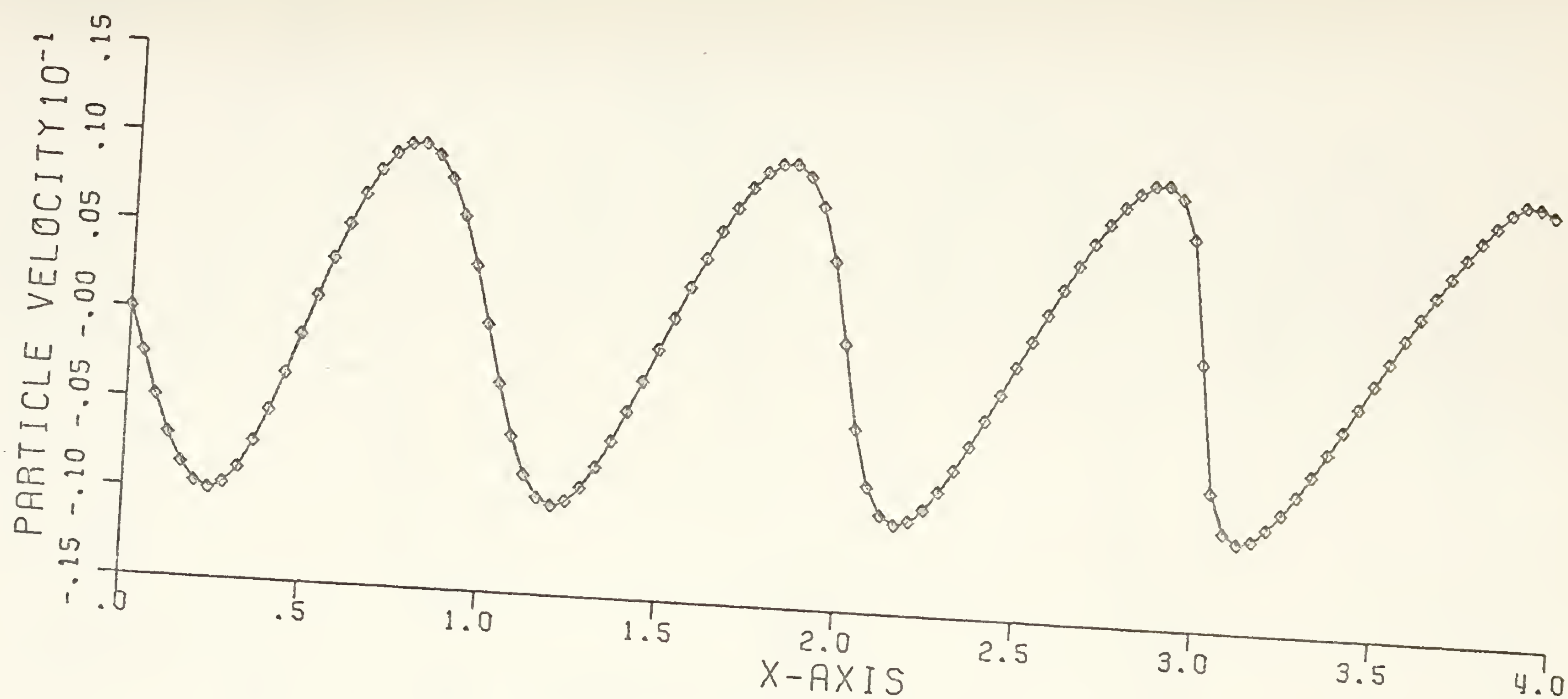


Figure 1a. Fubini Velocity Profile, $\beta\varepsilon = .04$, $\overline{X} \sim 4\lambda$

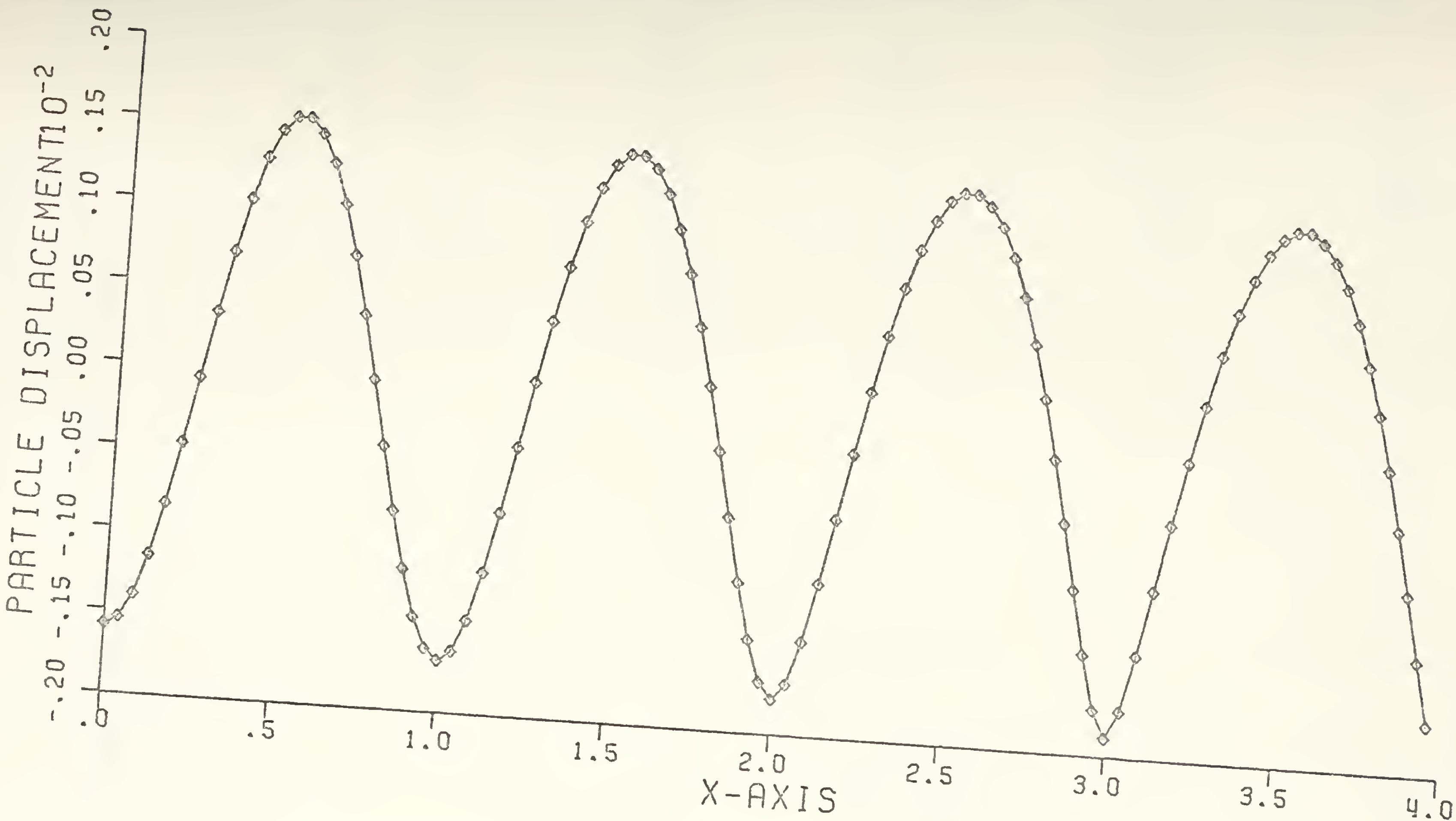


Figure 1b. Fubini Displacement Profile, $\beta\varepsilon = .04$, $\overline{X} \sim 4\lambda$

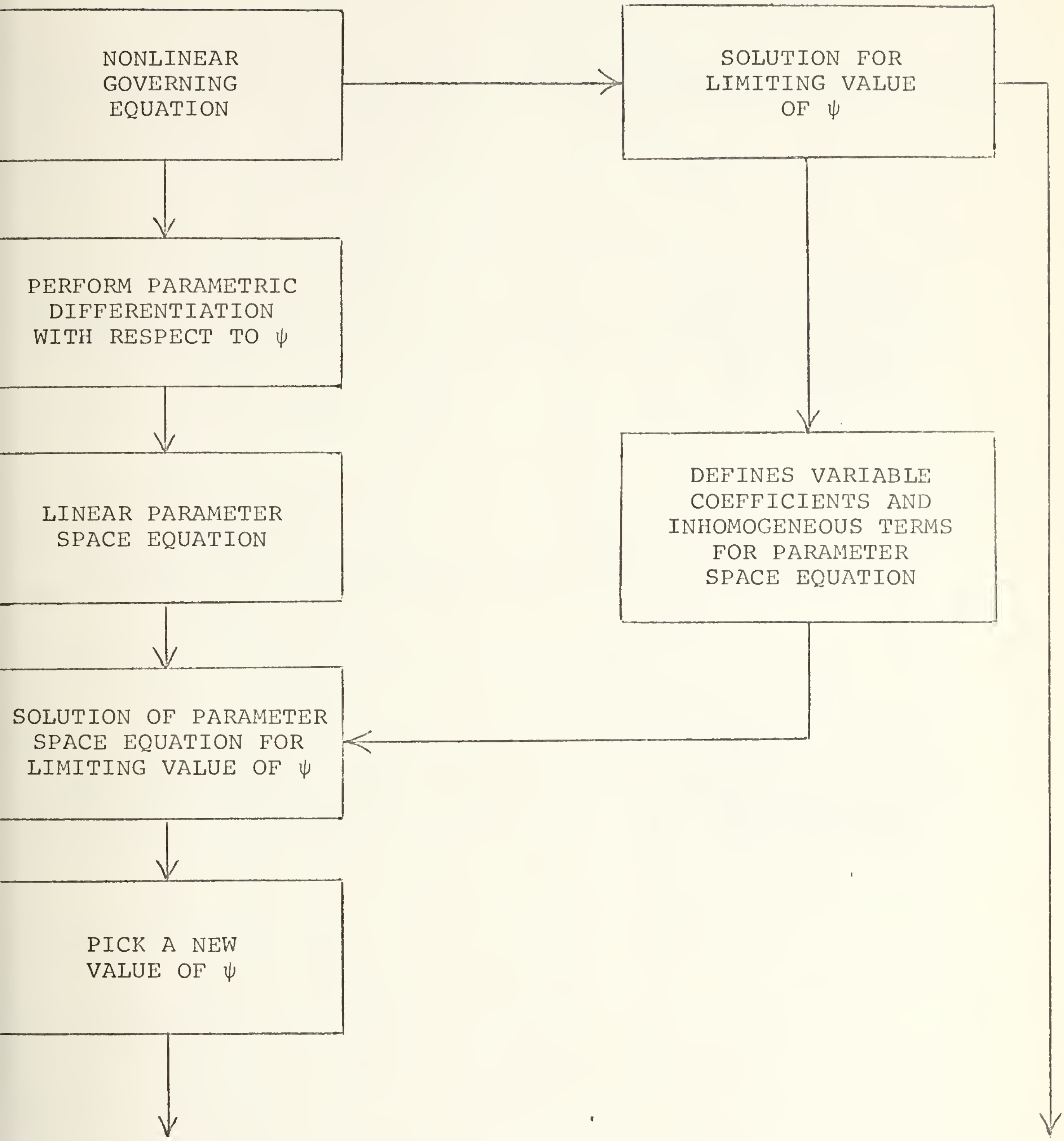


Figure 2 The Solution Procedure

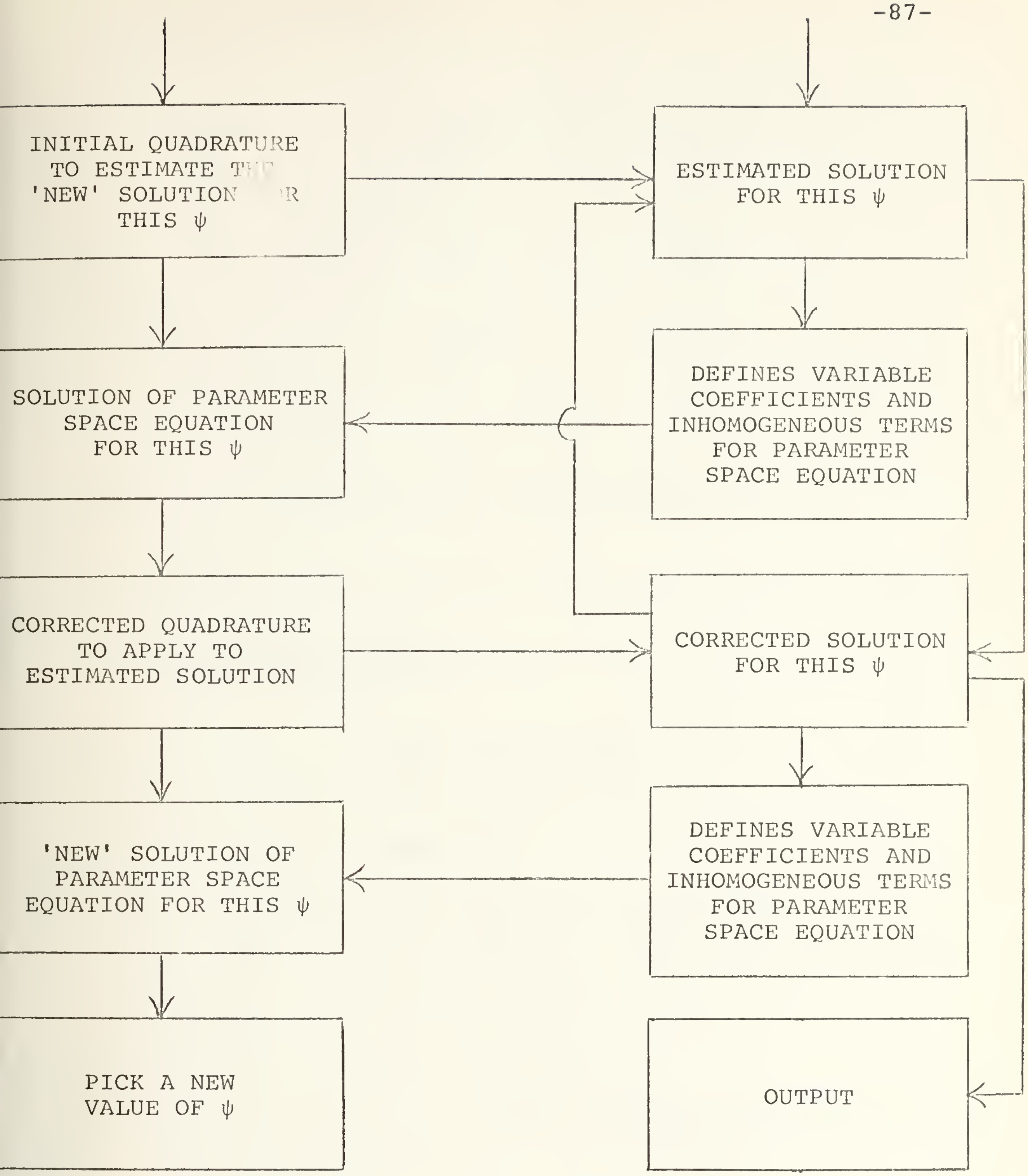


Figure 2 The Solution Procedure (cont.)

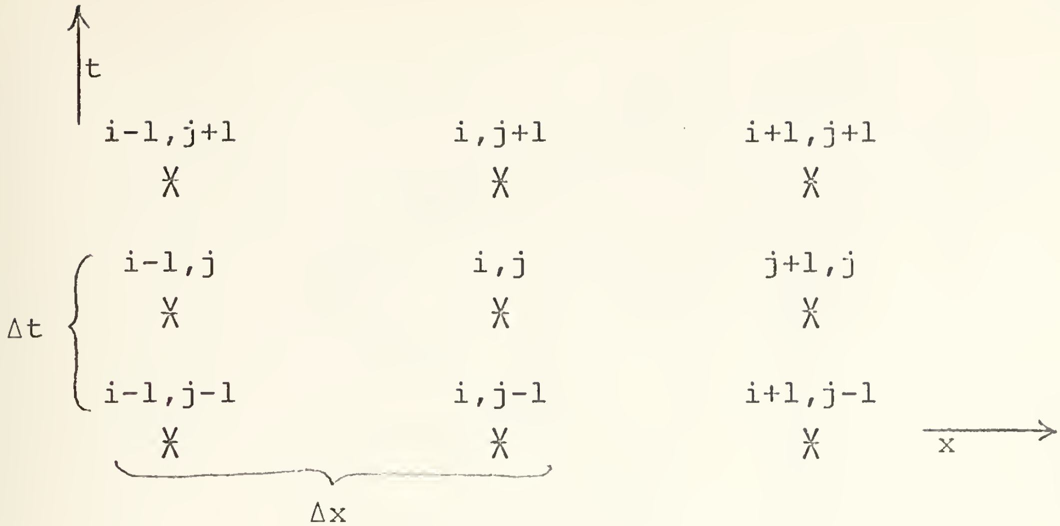
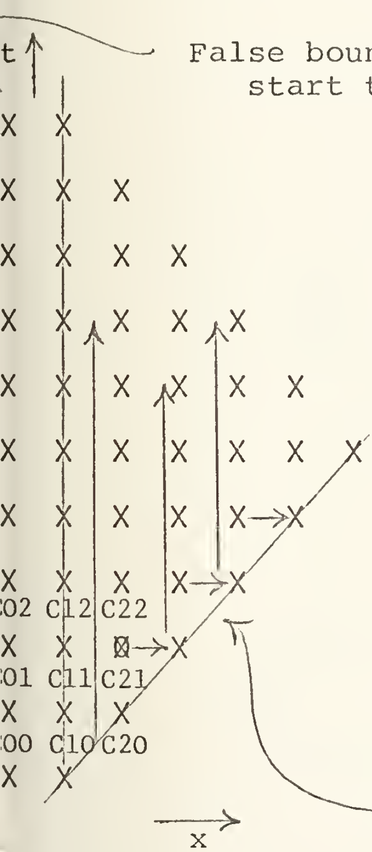


Figure 3 The Nine Point Finite Difference Cube

False boundary required to start the solution



$$C00 = -A_4/(\Delta x)^2\Delta t$$

$$C01 = A_1/(\Delta x)^2 + A_3/2\Delta x + A_4/(\Delta x)^2\Delta t$$

$$C02 = 0$$

$$C10 = -A_2/(\Delta t)^2 + 2A_4/(\Delta x)^2\Delta t$$

$$C11 = -2A_1/(\Delta x)^2 + 2A_2/(\Delta t)^2 - 2A_4/(\Delta x)^2\Delta t$$

$$C12 = -A_2/(\Delta t)^2$$

$$C20 = -A_4/(\Delta x)^2\Delta t$$

$$C21 = A_1/(\Delta x)^2 - A_3/2\Delta x + A_4/(\Delta x)^2\Delta t$$

$$C22 = 0$$

Points along this line must be estimated in order to calculate $g_{i+1,j}$ explicitly.

Figure 4 The Solution Mesh, the Arrows Indicate the Marching Direction Utilized Herein

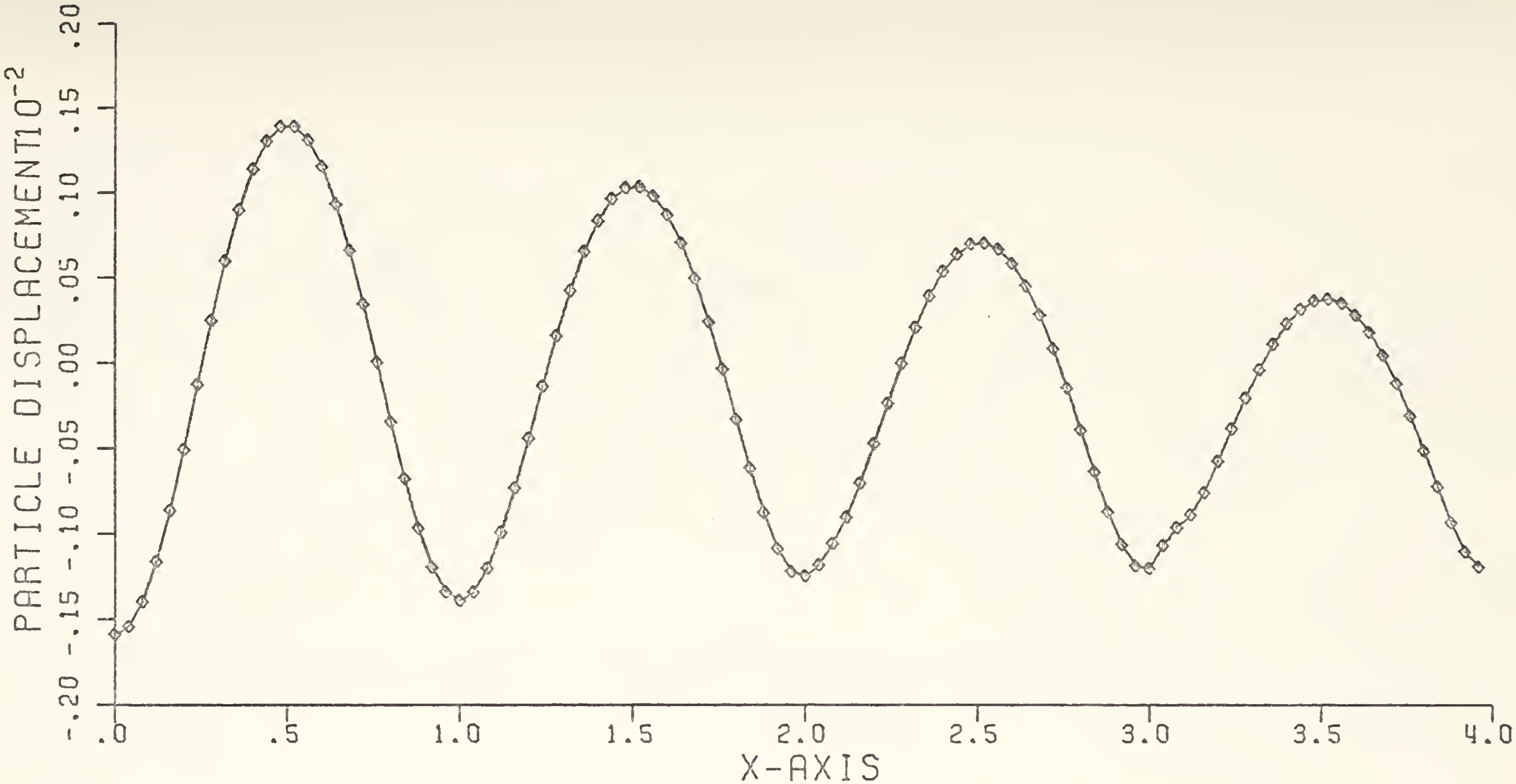


Figure 5a. Displacement Profile, $\beta\varepsilon$ = .04, Γ = 1.27

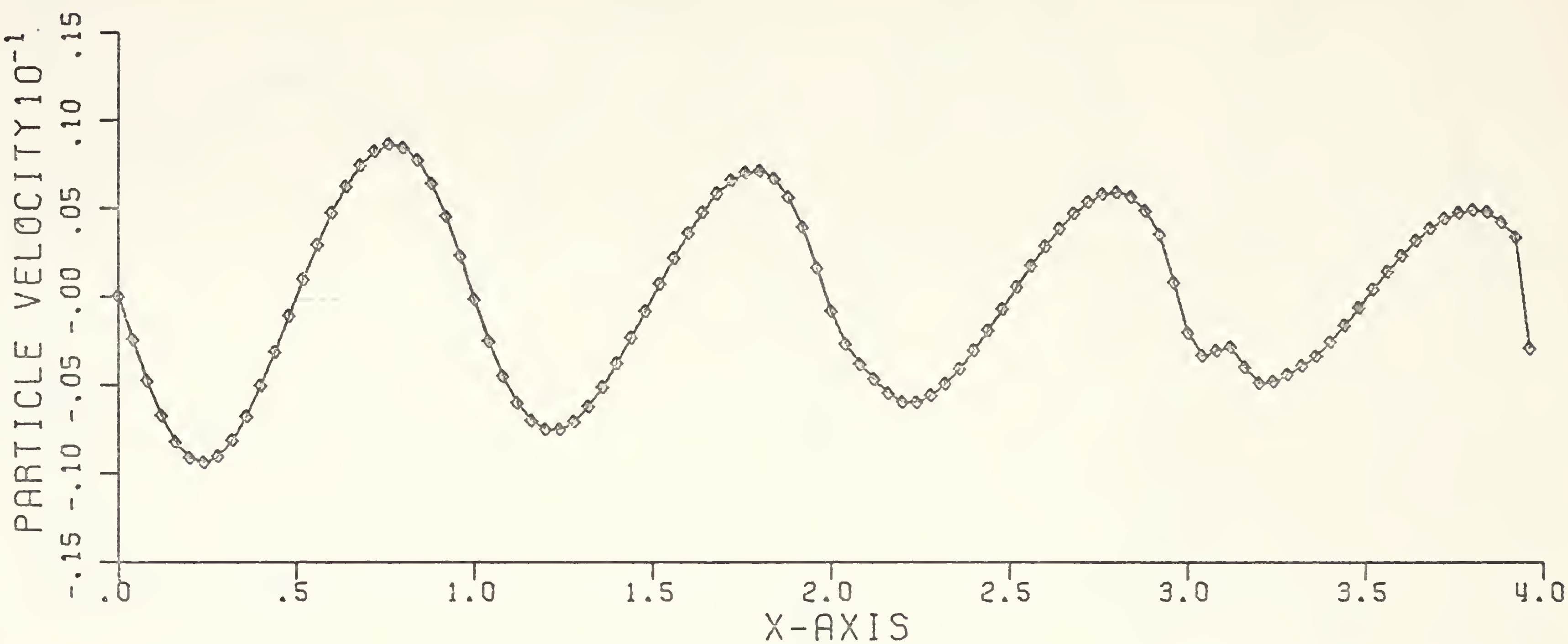


Figure 5b. Velocity Profile, $\beta\varepsilon$ = .04, Γ = 1.27

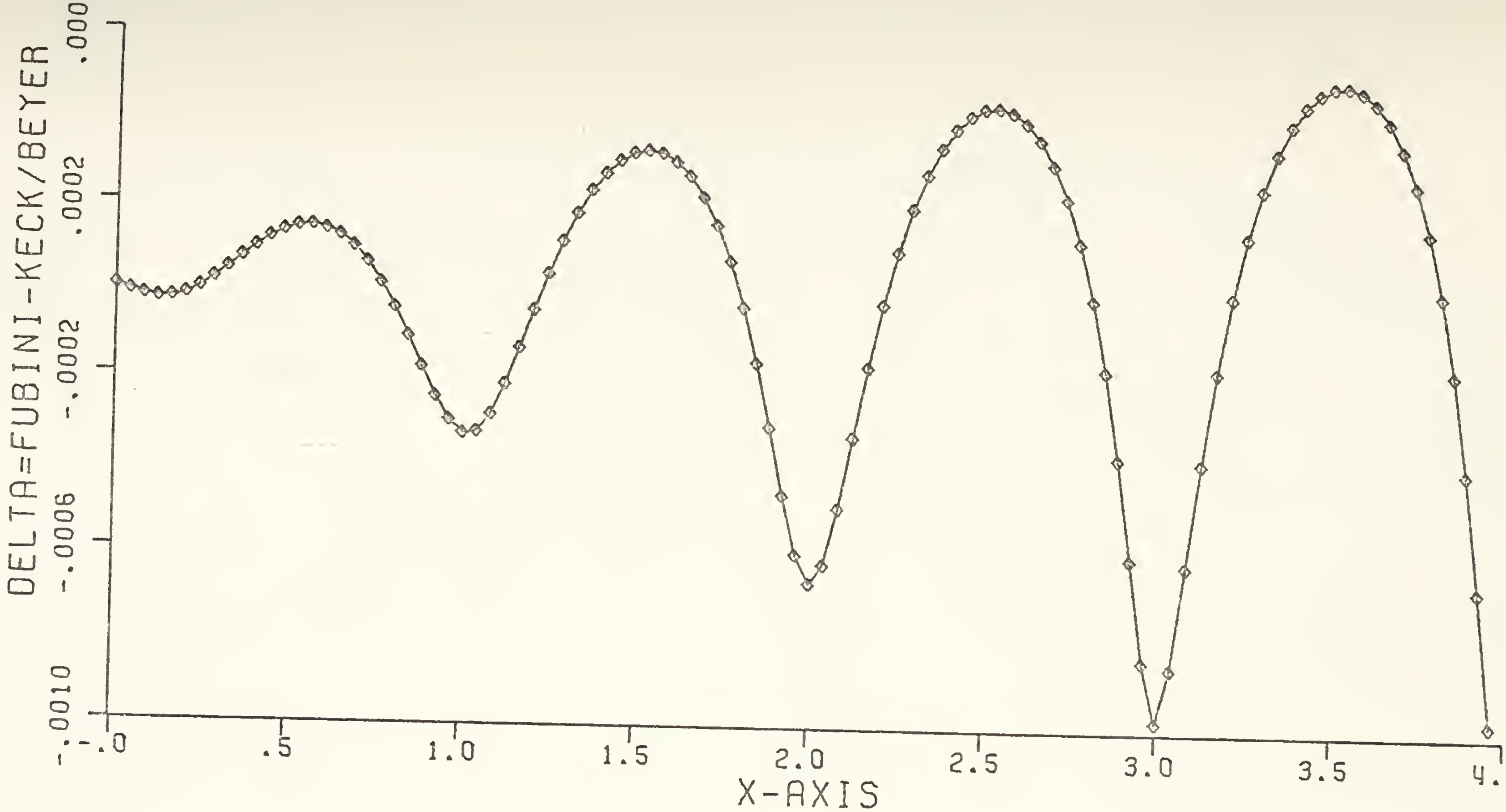


Figure 5c. Difference Between Fubini and Keck-Beyer Displacement Profiles
$\beta\varepsilon = .04,\ \Gamma = 1.27$

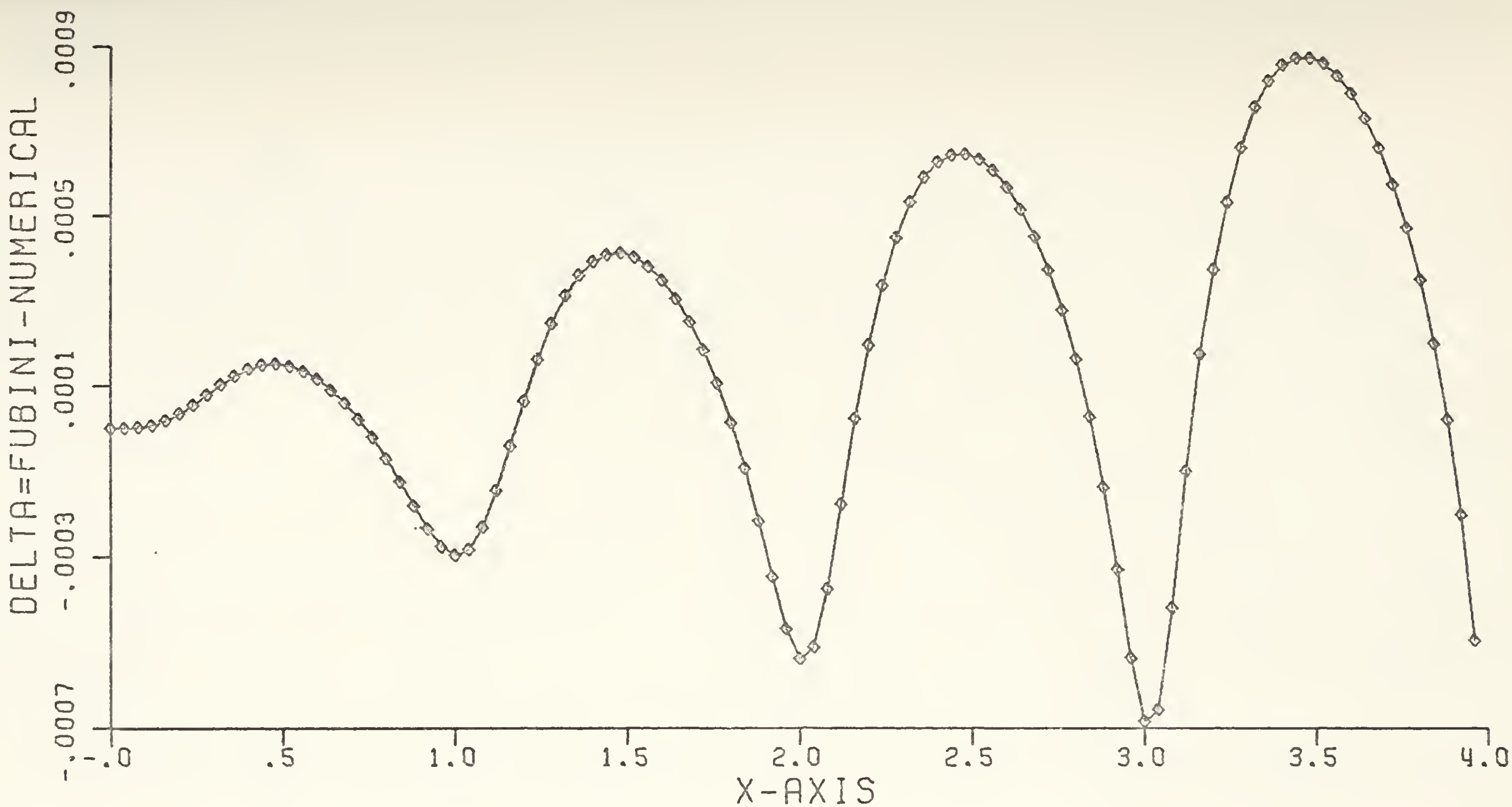


Figure 5d. Difference Between Fubini and P.D. Displacement Profiles
$\beta\varepsilon = .04$, $\Gamma = 1.27$

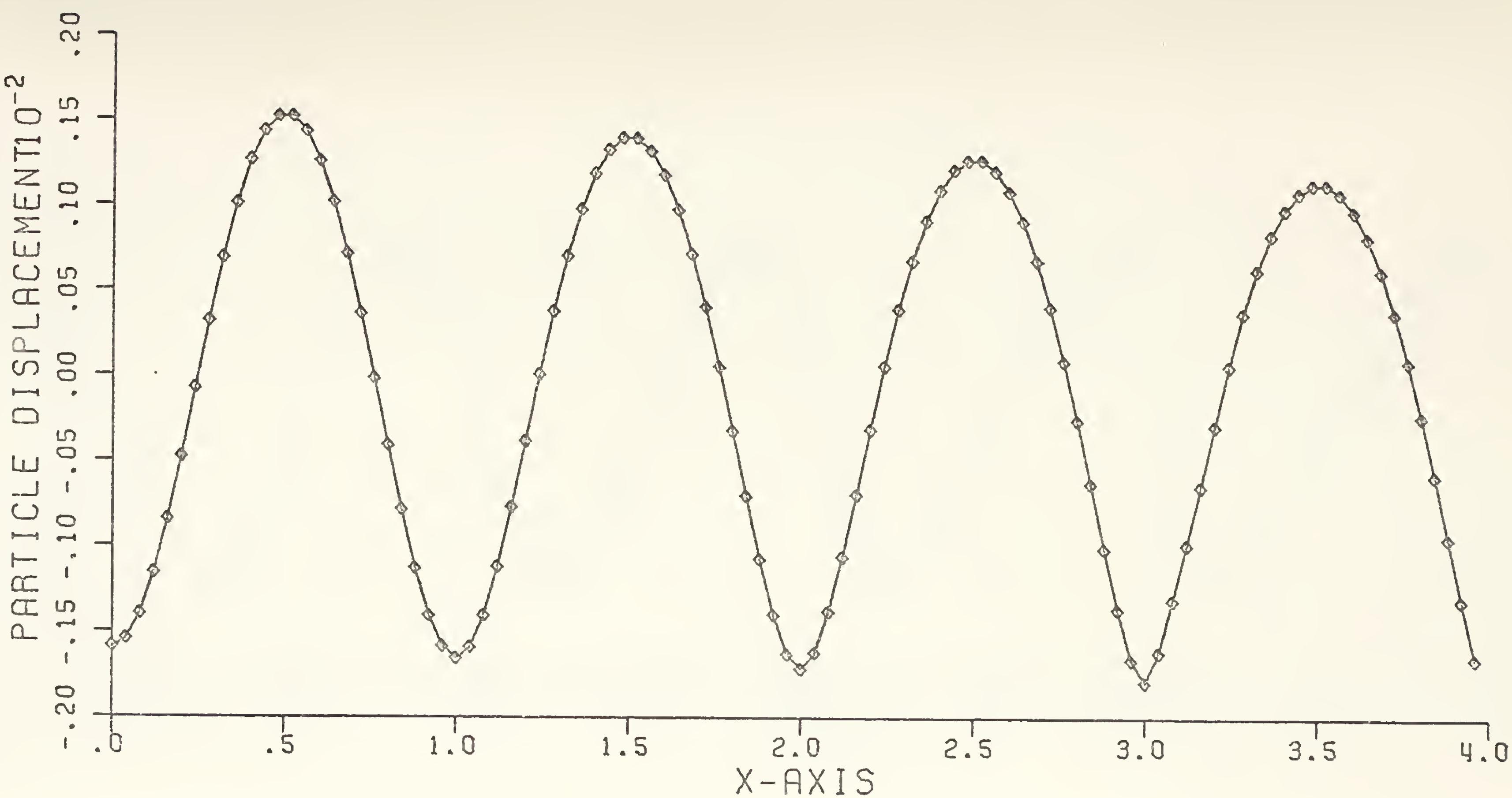


Figure 6a. Displacement Profile, $\beta\varepsilon = .04$, $\Gamma = 12.7$

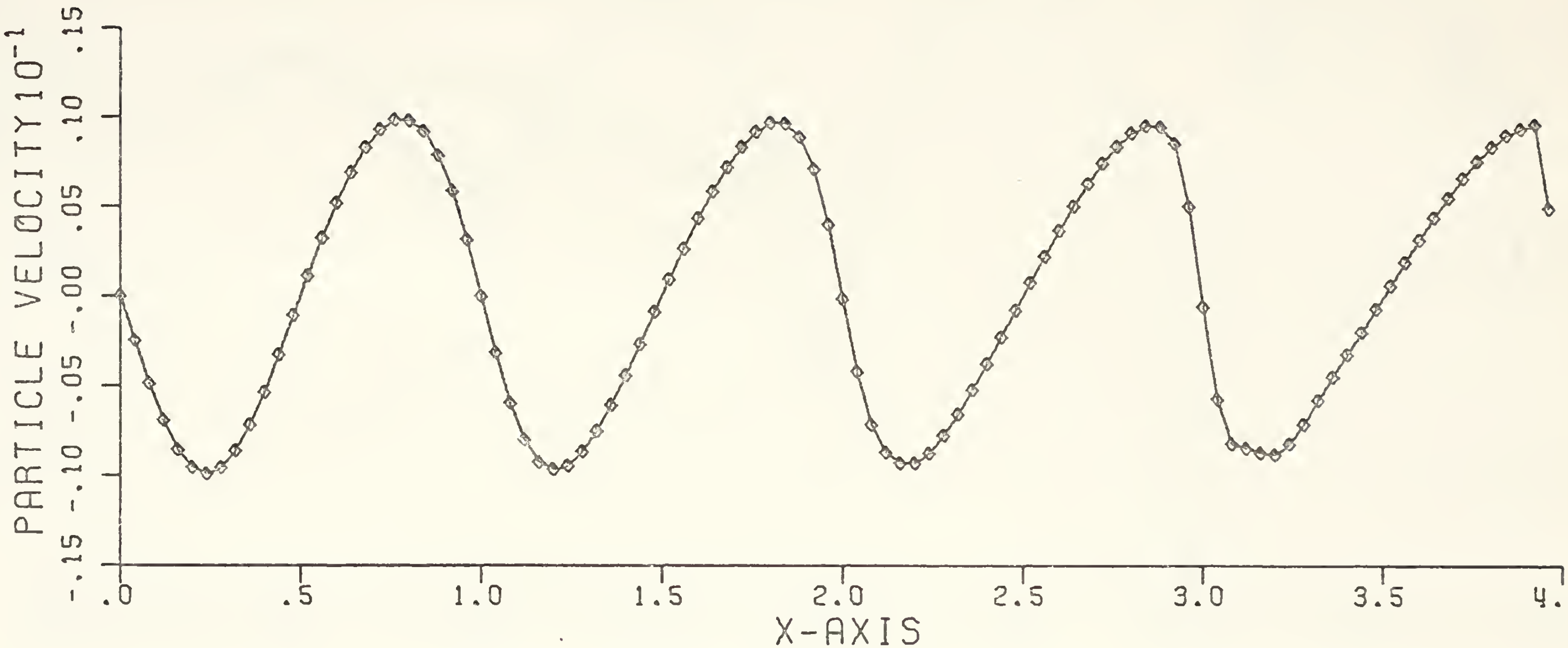


Figure 6b. Velocity Profile, $\beta\varepsilon$ = .04, Γ = 12.7

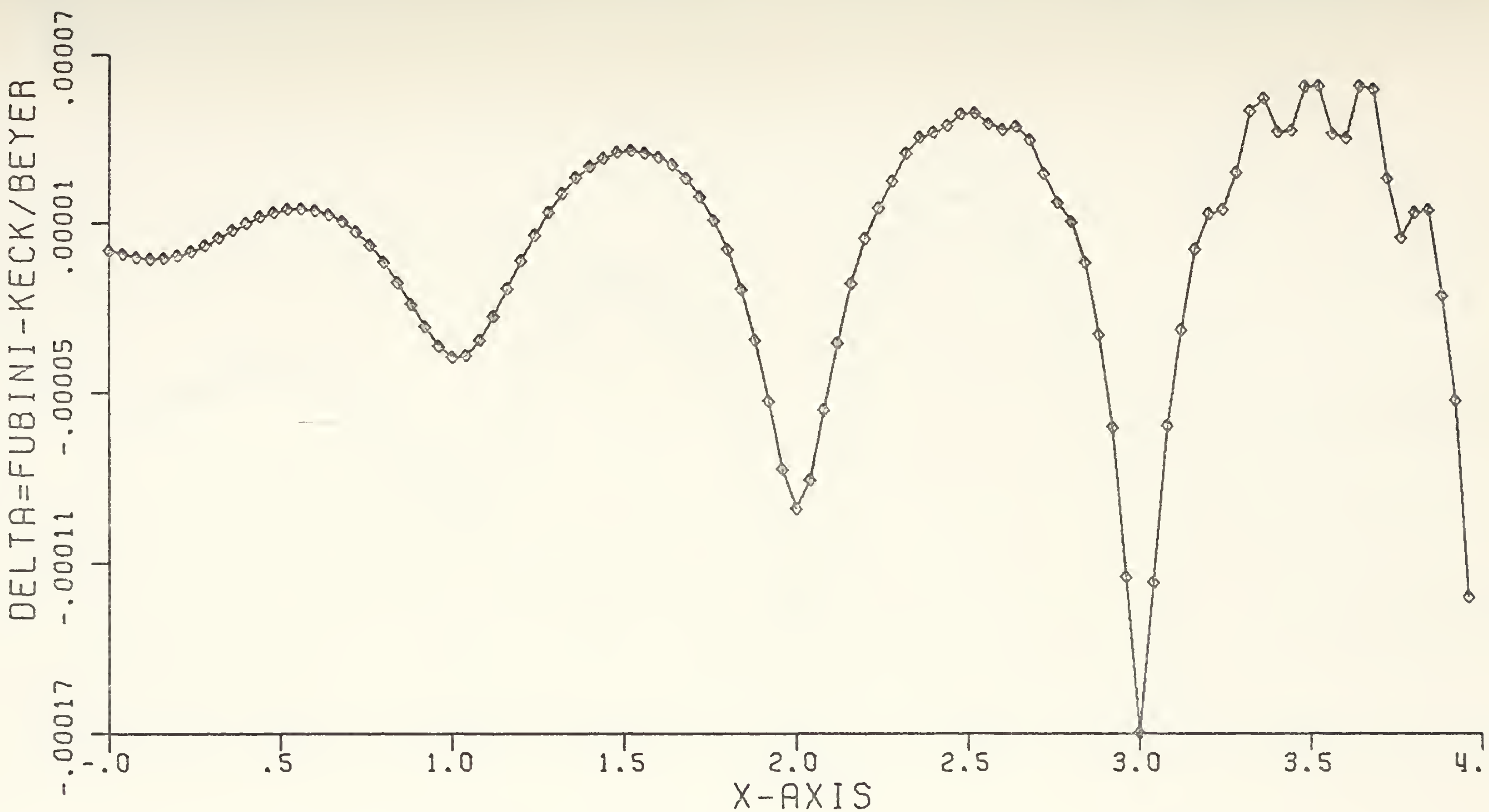


Figure 6c. Difference Between Fubini and Keck-Beyer Displacement Profiles
$\beta\varepsilon = .04,\ \Gamma = 12.7$

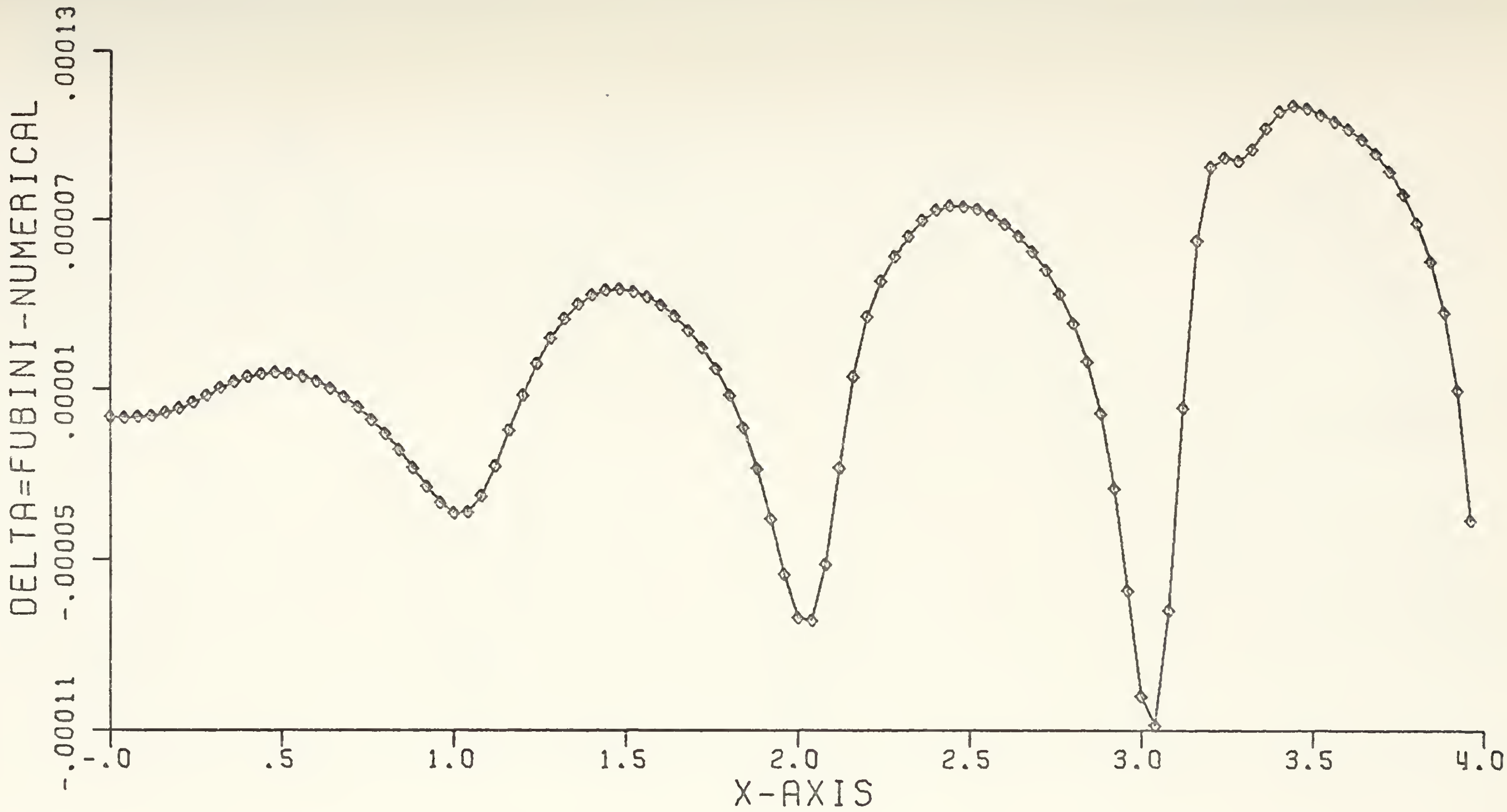


Figure 6d. Difference Between Fubini and P.D. Displacement Profiles
$\beta\varepsilon = .04$, $\Gamma = 12.7$

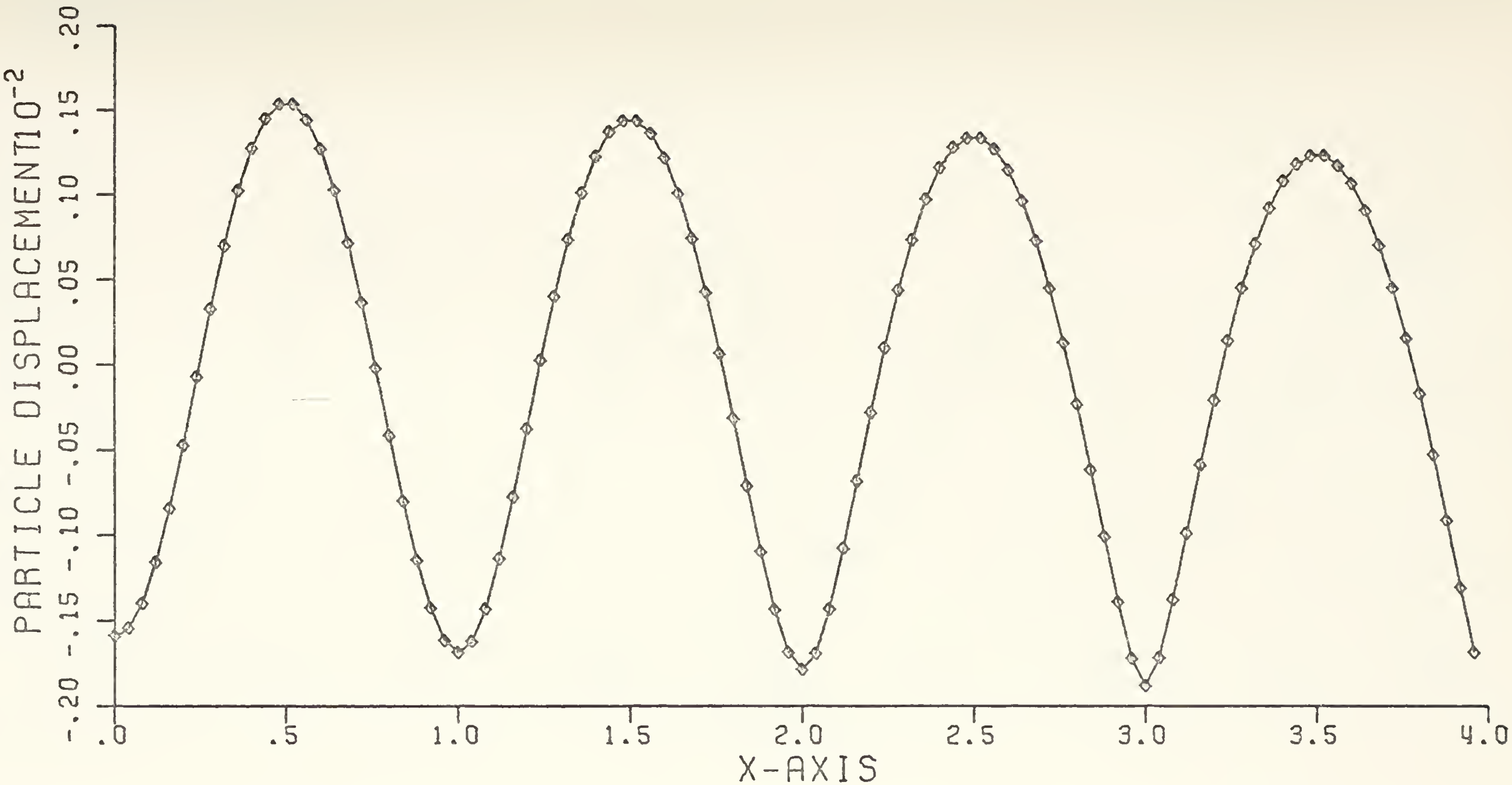


Figure 7a. Displacement Profile, $\beta\varepsilon = .04$, $\Gamma = 275$

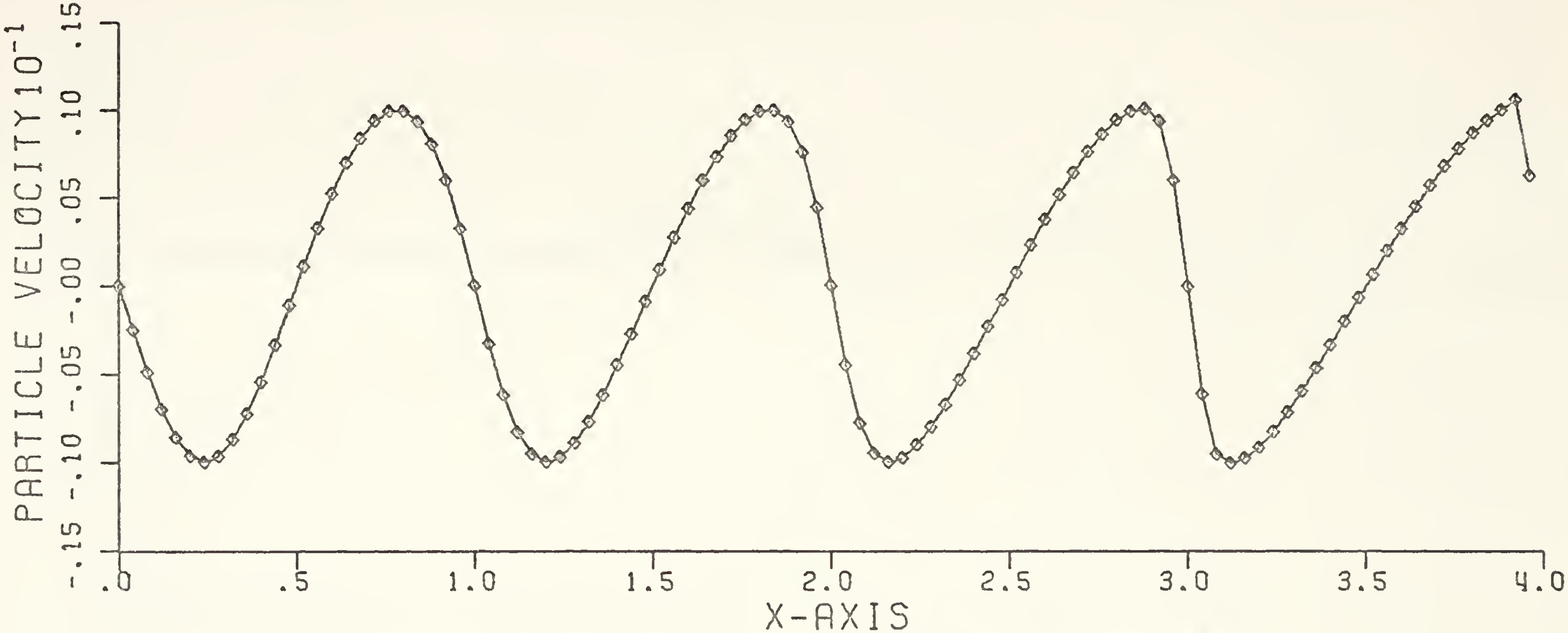


Figure 7b. Velocity Profile, $\beta\varepsilon$ = .04, Γ = 275

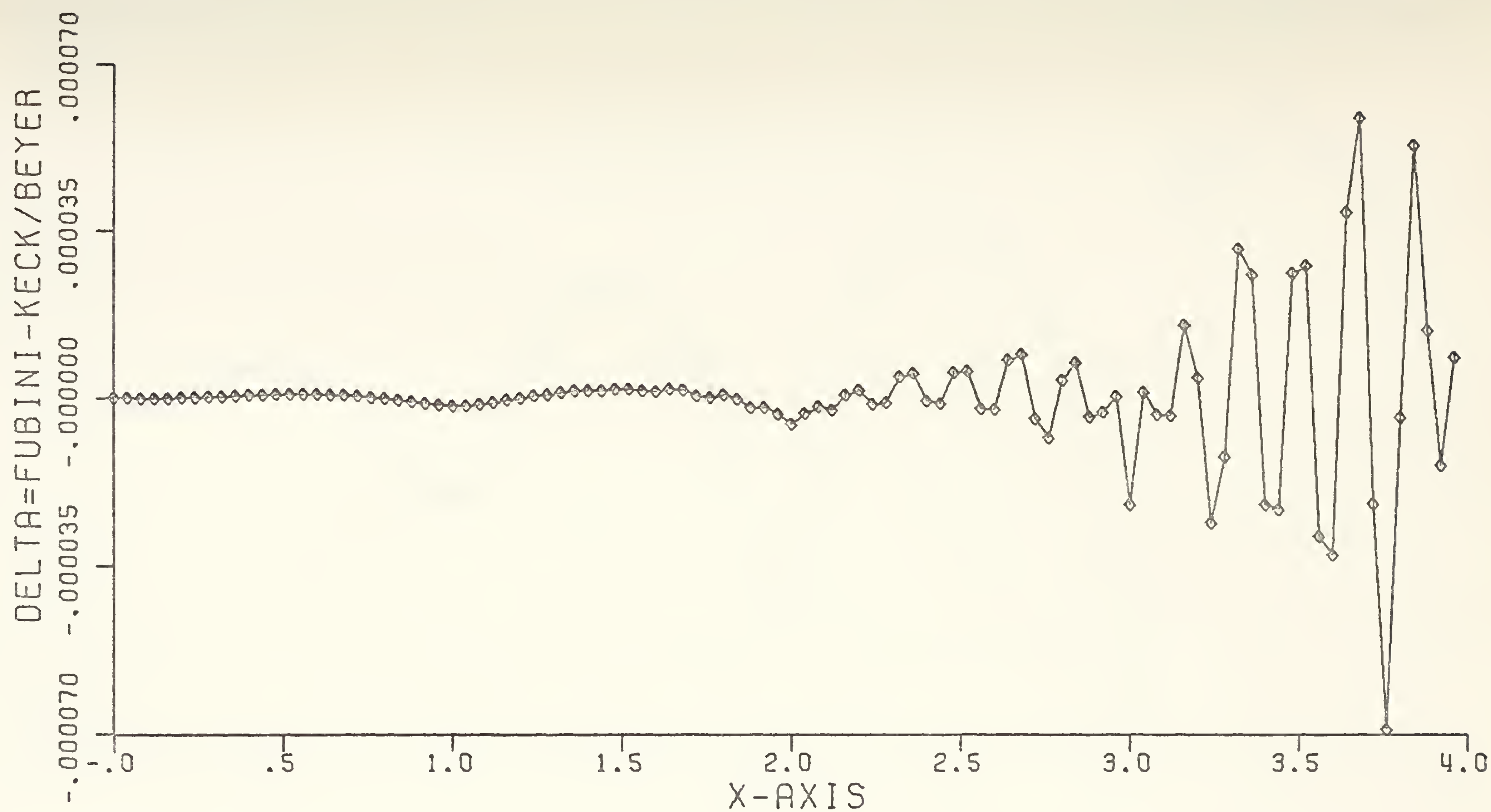


Figure 7c. Difference Between Fubini and Keck-Beyer Displacement Profiles
$\beta\varepsilon = .04,\ \Gamma = 275$

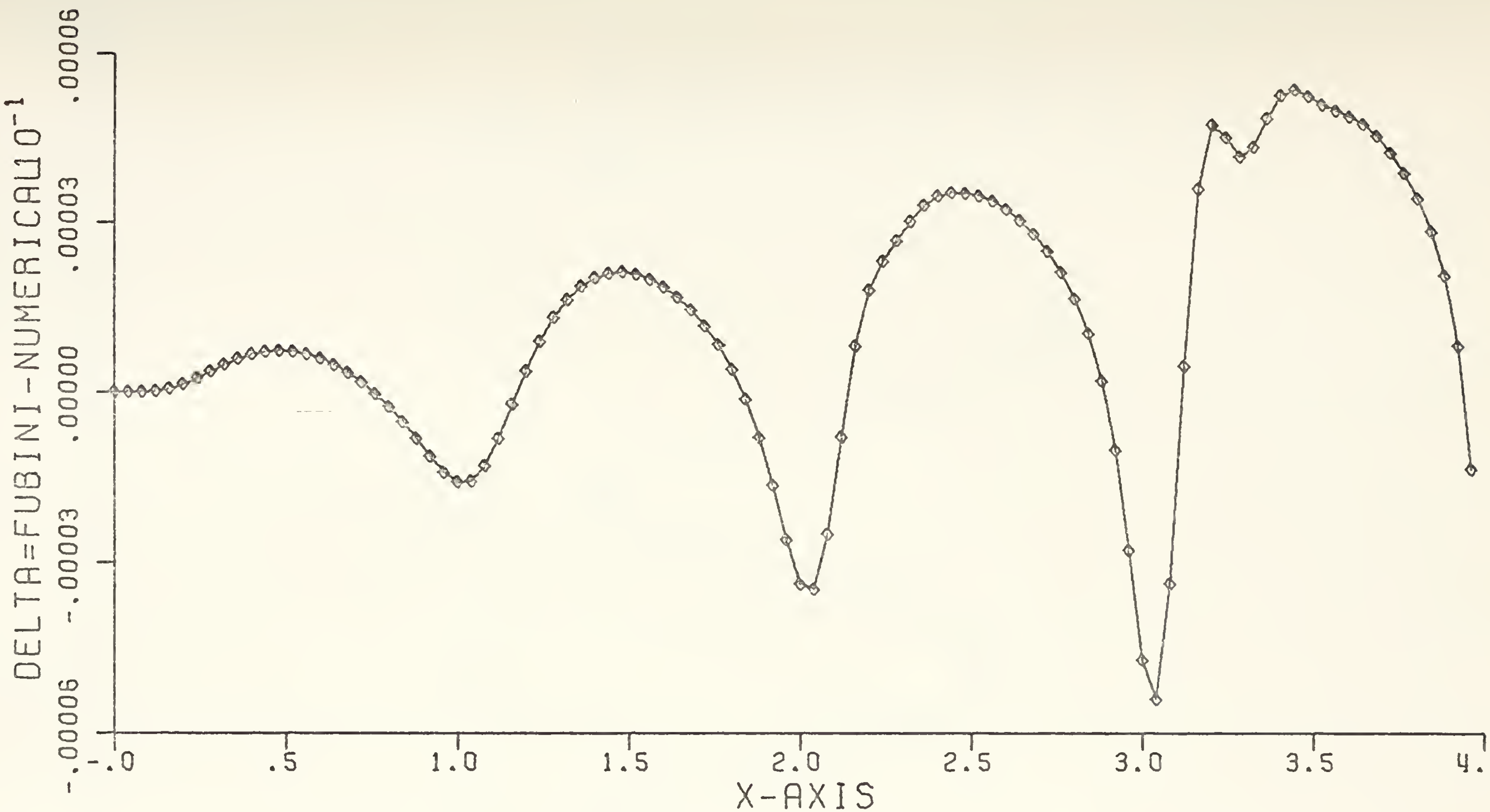


Figure 7d. Difference Between Fubini and P.D. Displacement Profiles
$\beta\varepsilon = .04$, $\Gamma = 275$

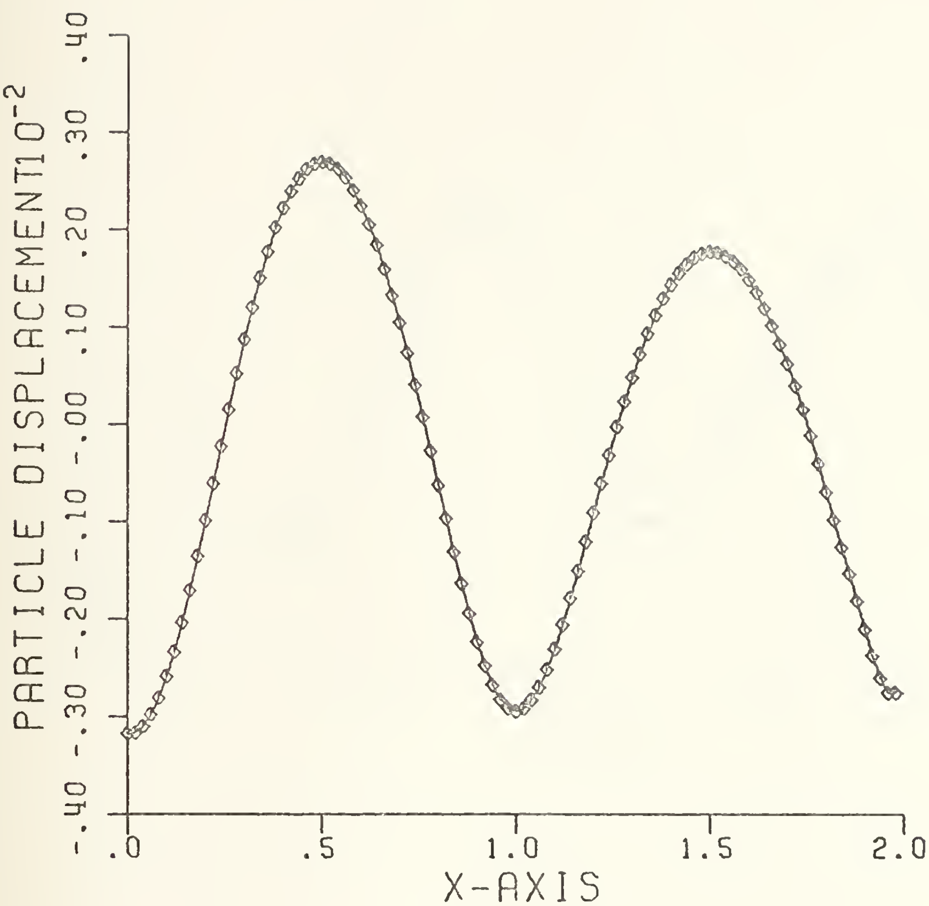


Figure 8a. Displacement Profile, $\beta\varepsilon$ = .08, Γ = 2.55

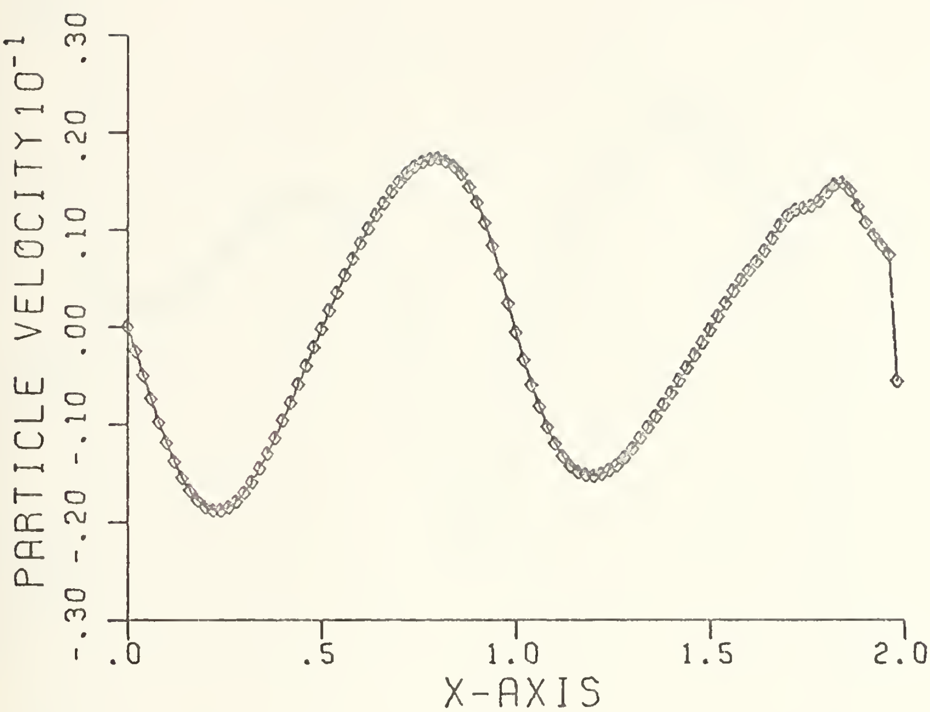


Figure 8b. Velocity Profile, $\beta\varepsilon$ = .08, Γ = 2.55

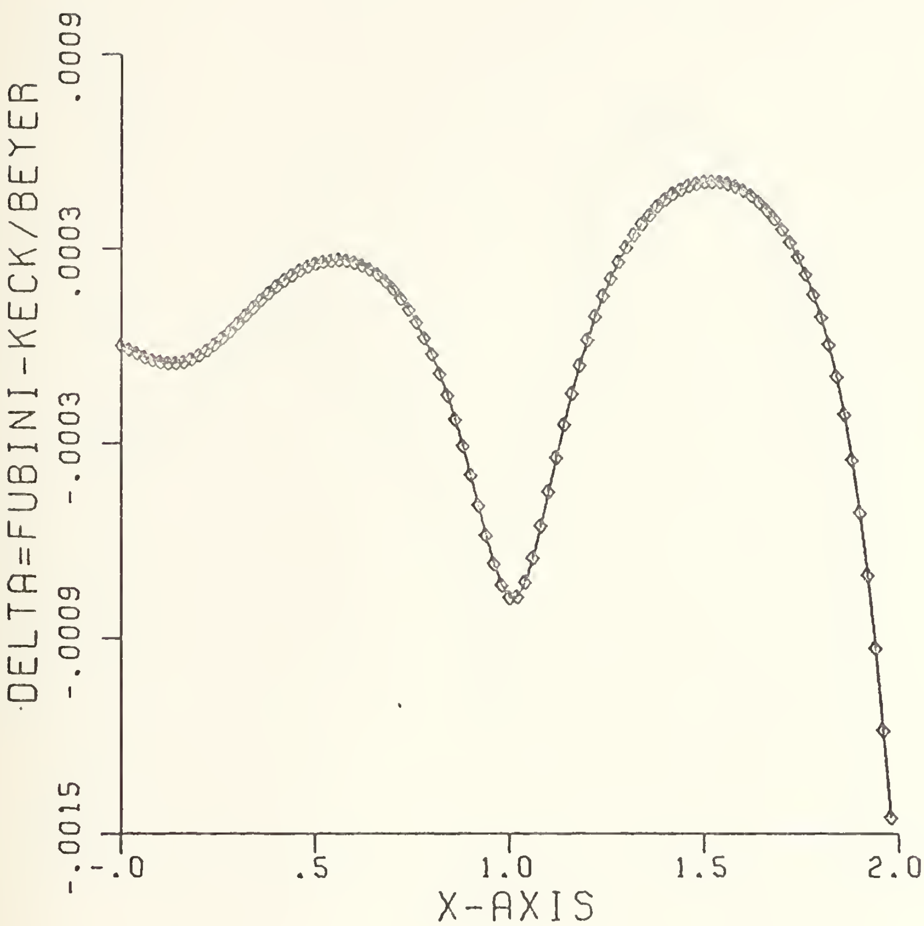


Figure 8c. Difference Between Fubini and Keck-Beyer Displacement Profiles, $\beta\varepsilon$ = .08, Γ = 2.55

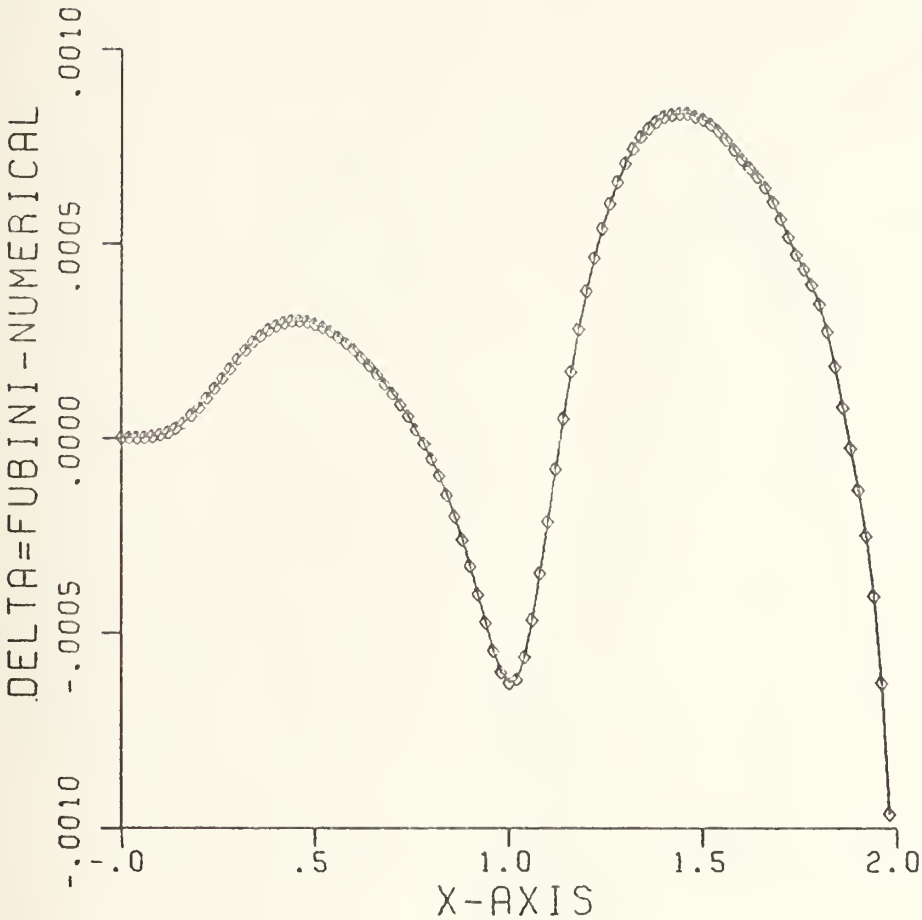


Figure 8d. Difference Between Fubini and P.D. Displacement Profiles, $\beta\varepsilon$ = .08, Γ = 2.55

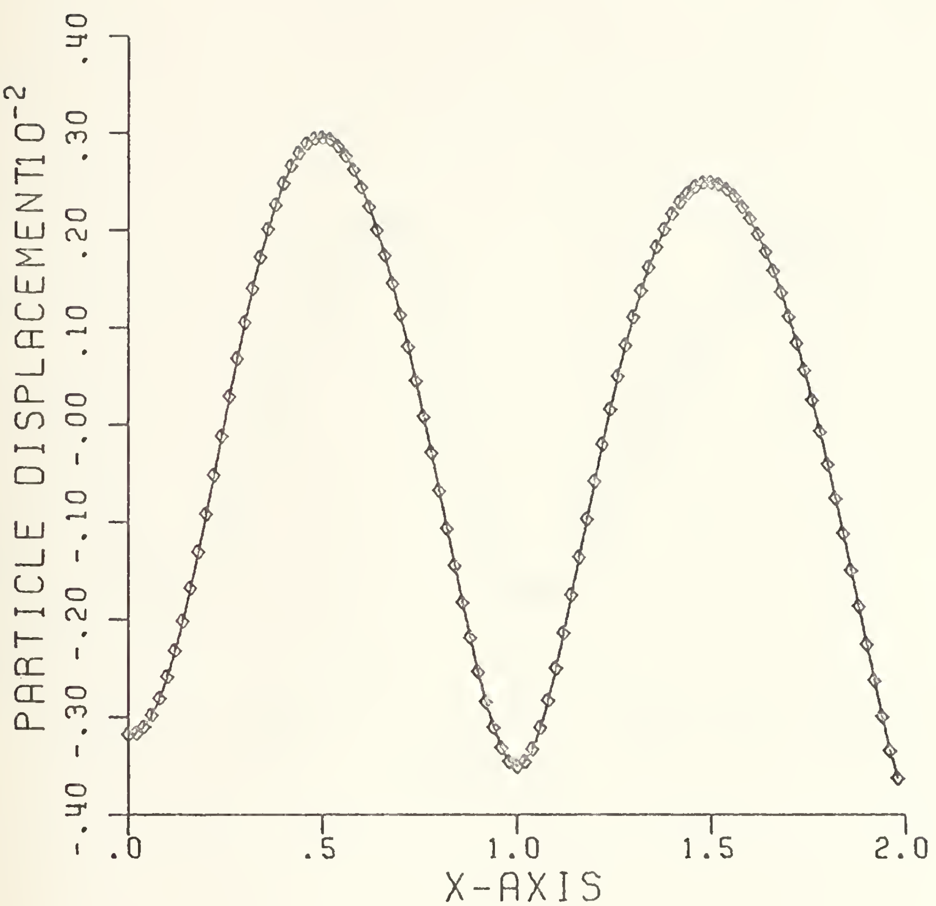


Figure 9a. Displacement Profile, $\beta\varepsilon$ = .08, Γ = 25.5

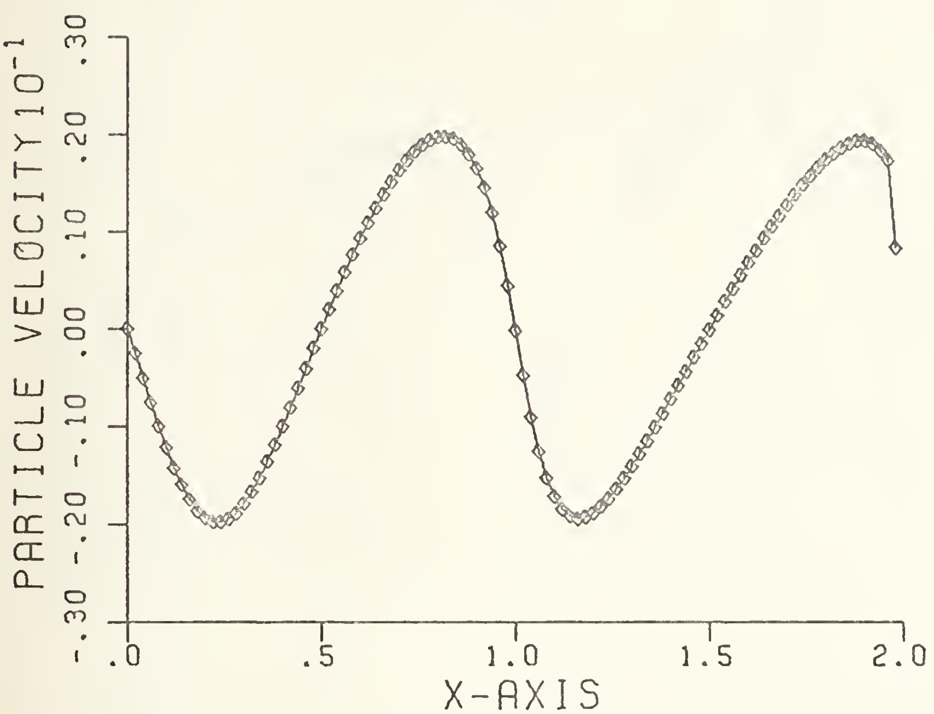


Figure 9b. Velocity Profile, $\beta\varepsilon$ = .08, Γ = 25.5

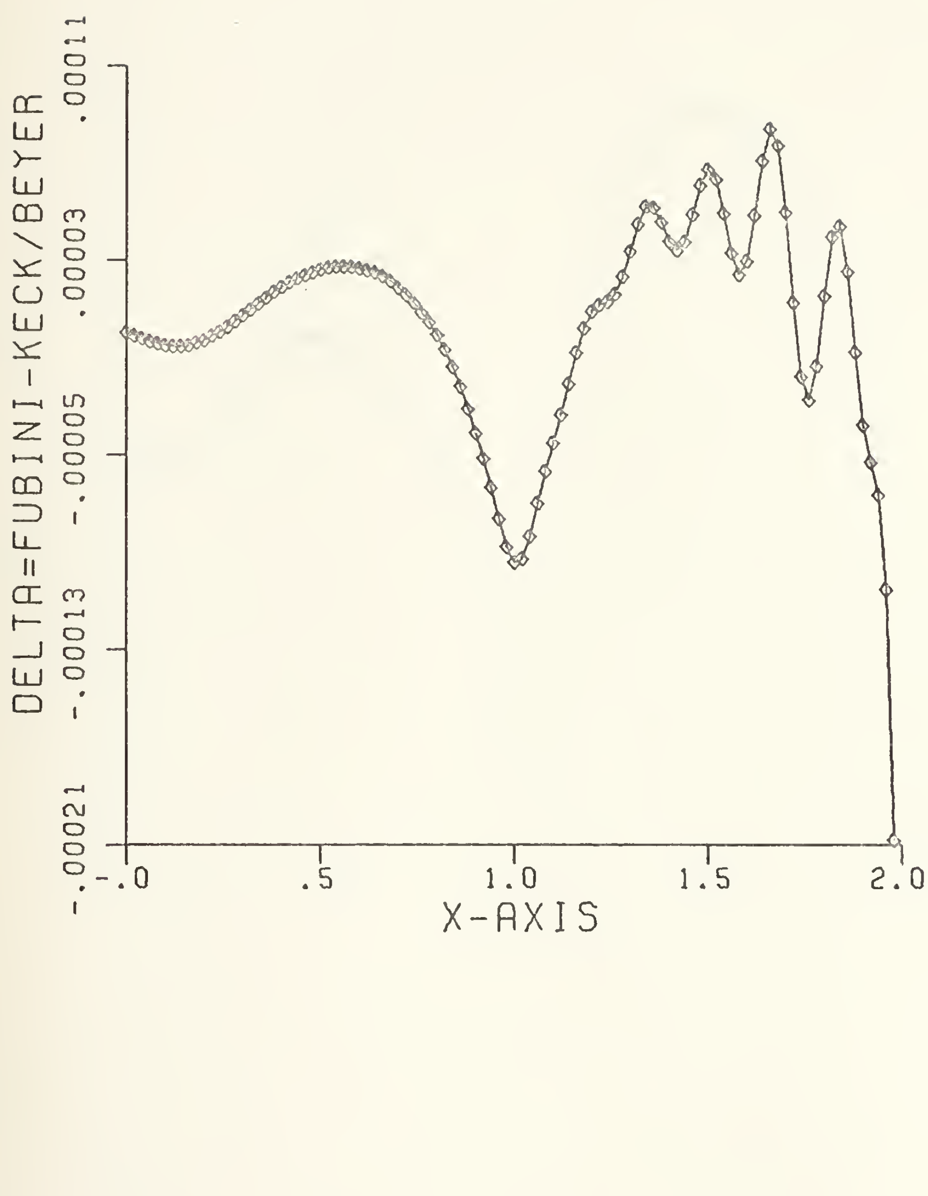


Figure 9c. Difference Between Fubini and Keck-Beyer Displacement Profiles, $\beta\varepsilon$ = .08, Γ = 25.5

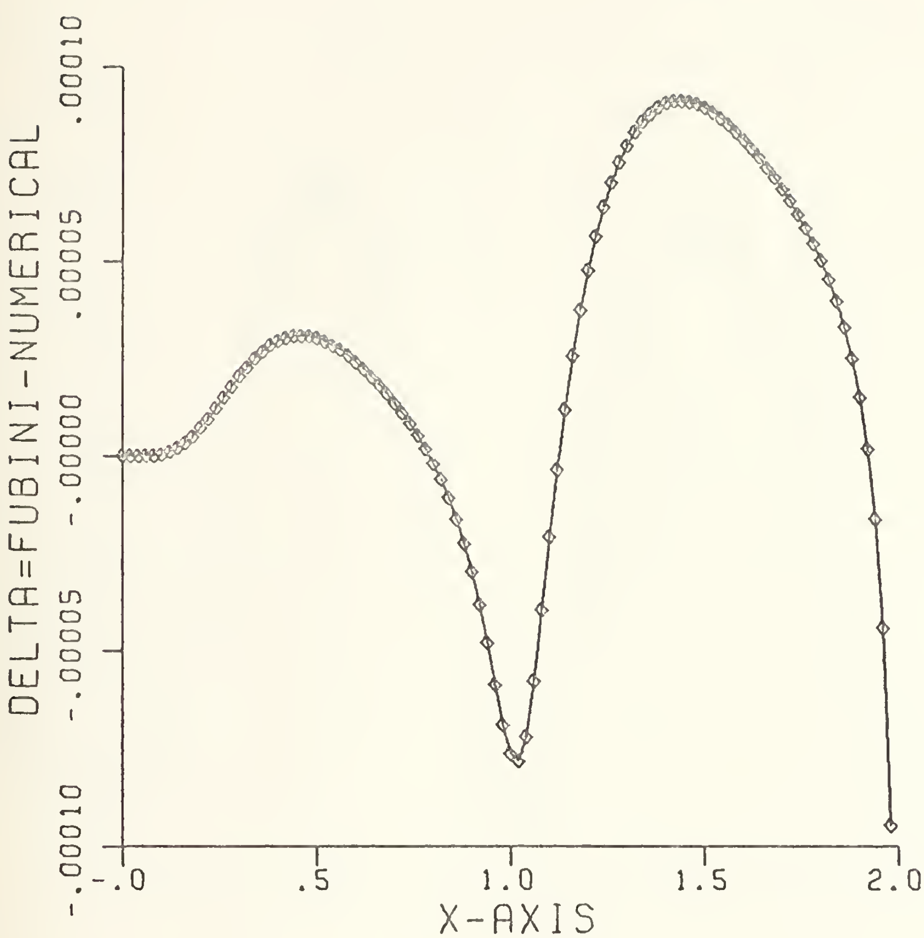


Figure 9d. Difference Between Fubini and P.D. Displacement Profiles, $\beta\varepsilon = .08$, $\Gamma = 25.5$

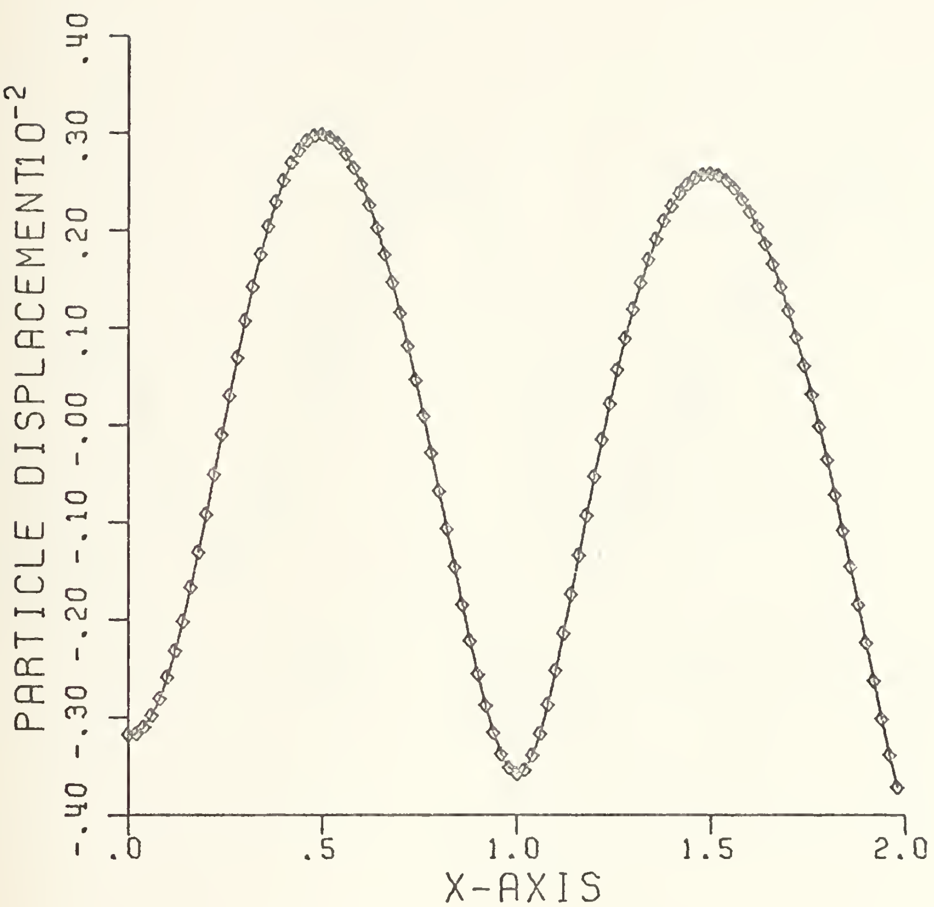


Figure 10a. Displacement Profile, $\beta\varepsilon$ = .08, Γ = 550

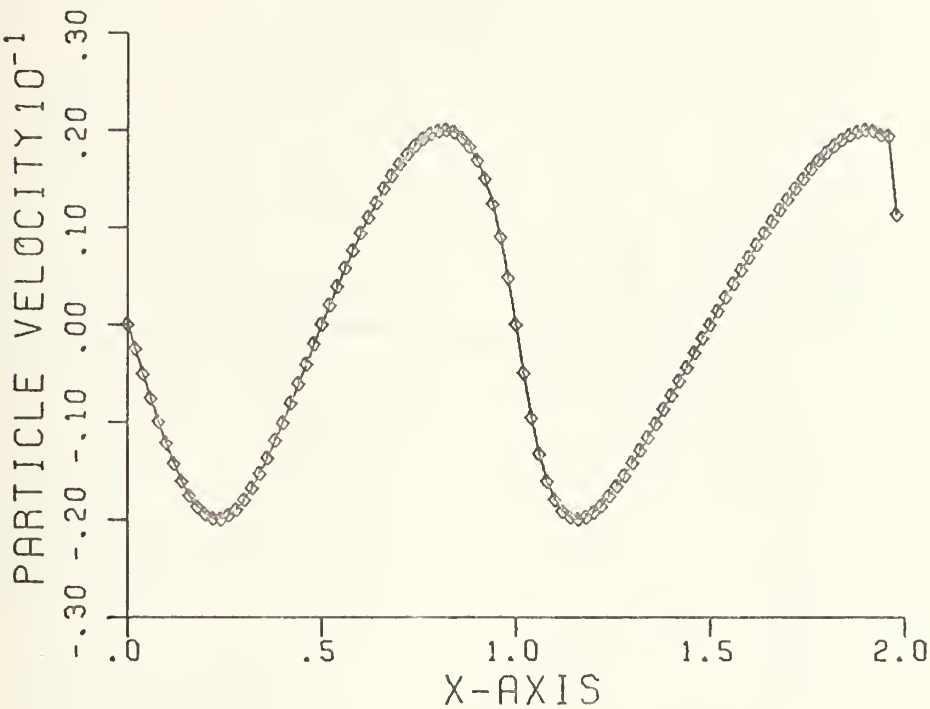


Figure 10b. Velocity Profile, $\beta\varepsilon$ = .08, Γ = 550

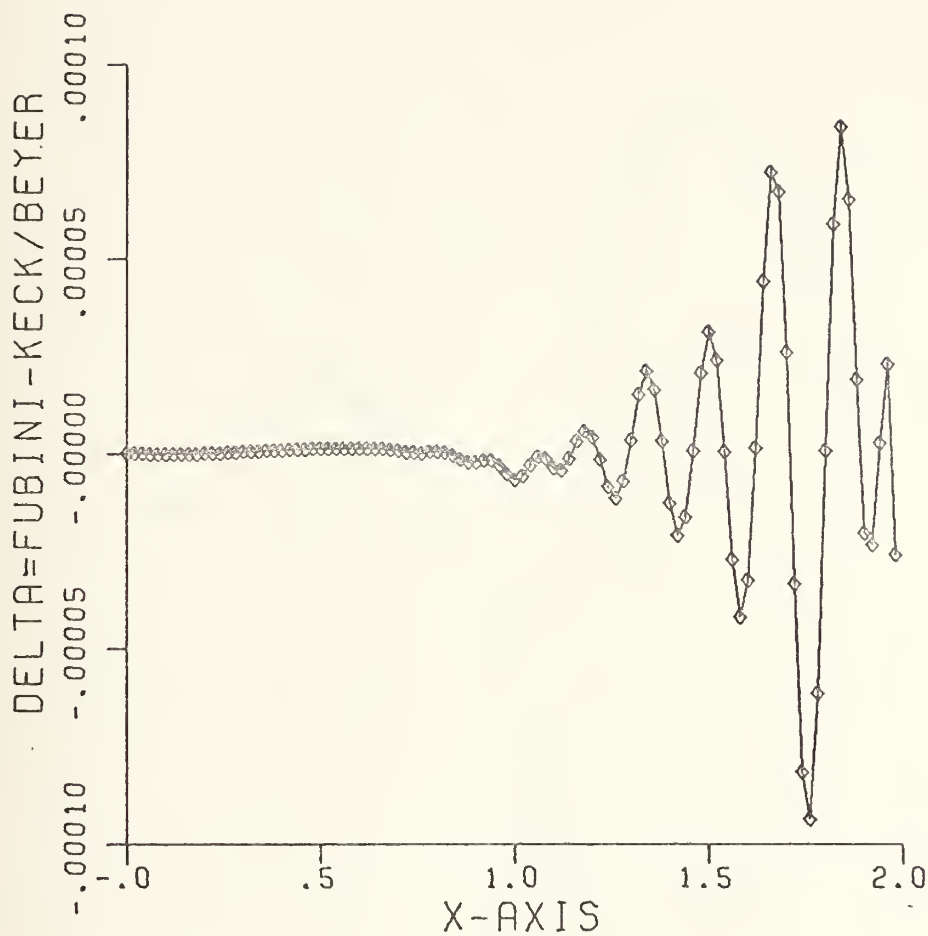


Figure 10c. Difference Between Fubini and Keck-Beyer Displacement Profiles, $\beta\varepsilon$ = .08, Γ = 550

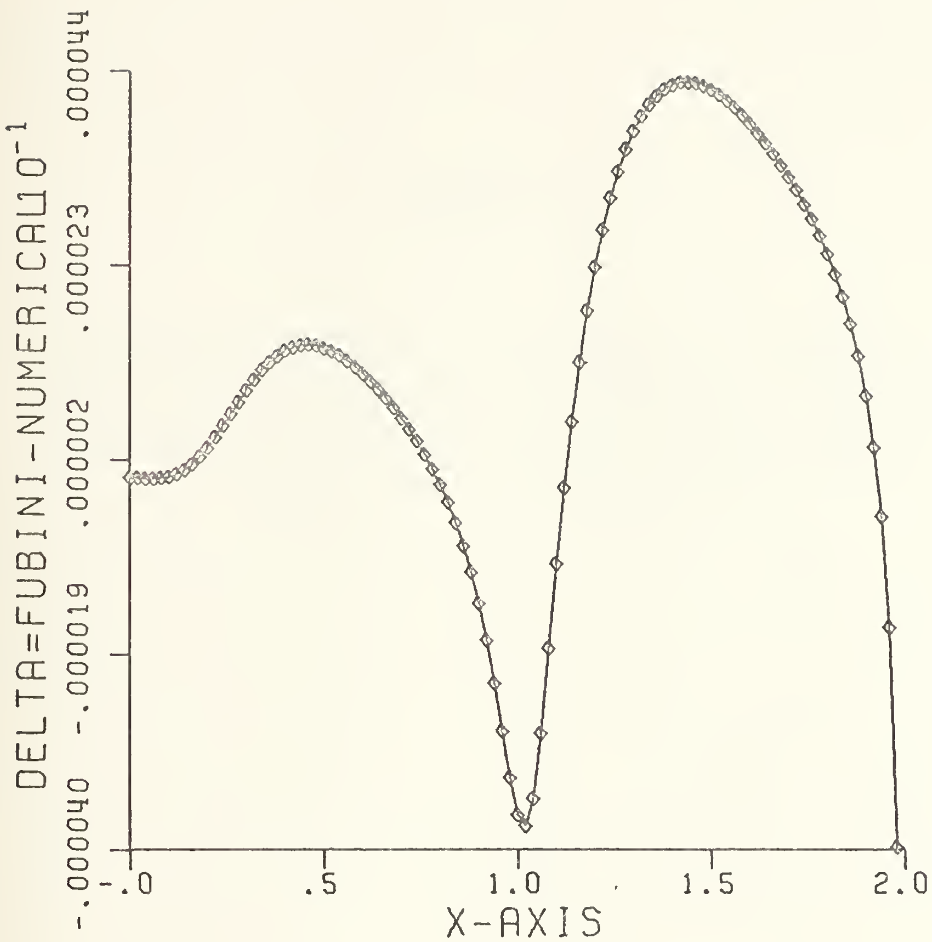


Figure 10d. Difference Between Fubini and P.D. Displacement Profiles, $\beta\varepsilon$ = .08, Γ = 550

APPENDIX A

A Computer Program for Solution and Analysis of the Nonlinear Plane Wave Propagation Problem in a Viscous Fluid Utilizing the Technique of Parametric Differentiation

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C  PROGRAM FOR SOLUTION OF NON-LINEAR PLANE WAVE PROPAGATION PROBLEM IN A
C   VISCOUS FLUID BY THE TECHNIQUE OF PARAMETRIC DIFFERENTIATION
      DIMENSION FUB(10500),GIN(10500),SUM(105),DIFF(105),OUT(105),FI(105
     100),S(10500),TH(10500),Y(105),Z(105),V(105),U(210),YU(105),ZU(105)
     2,VU(105),ALPHA(50),PV(105),GOUT(10500),DIFFS(105),VFUB(105)
C MAIN VARIABLES
C      FUB--ARRAY OF SOLUTION VALUES FOR DISPLACEMENT
C      GIN,GOUT--ARRAYS OF THE SOLUTION OF THE PARAMETER SPACE EQUATION,GIN FOR
C                 THE PAST VALUE OF THE PARAMETER,GOUT FOR THE CURRENT VALUE
C      SUM--A WORKING VARIABLE FOR THE CALCULATION OF THE INVISCID SOLUTION
C      DIFF,DIFFS--DIFFS IS A VECTOR OF SELECTED VALUES OF THE INVISCID SOLUTION
                   USED TO CALCULATE DIFF,A VECTOR OF SELECTED VALUES OF THE
C                   DIFFERENCE BETWEEN THE INVISCID AND P.D. SOLUTIONS
C      OUT-- A VECTOR OF SELECTED VALUES OF THE P.D. SOLUTION
C      FI--AN ARRAY OF VALUES OF THE FIRST PARTIAL OF FUB WITH RESPECT TO X
C      S--AN ARRAY OF THE VALUES OF THE SECOND PARTIAL OF FUB WITH RESPECT TO X
C      TH--AN ARRAY OF THE VALUES OF THE FIRST PARTIAL OF S WITH RESPECT TO T
C      Y,Z,V,U,YU,ZU,VU--WORKING VARIABLES FOR USE IN SUBROUTINES WHICH CALCULATE
C                        THE PARTIAL DERIVATIVES
C      ALPHA--A VECTOR OF VALUES OF THE PARAMETER FOR WHICH THE EQUATIONS ARE TO
C             BE SOLVED
C      PV-- A VECTOR OF SELECTED VALUES OF PARTICLE VELOCITY YIELDED BY THE P.D.
C           SOLUTION
C      VFUB--A VECTOR OF SELECTED VALUES OF THE FUBINI VELOCITY SOLUTION
C      X,T--SPACE AND TIME VARIABLES
C      BETA--PARAMETER OF NON-LINEARITY
C      EPS--MACH NUMBER
C      DELX,DELT--SPATIAL AND TEMPORAL STEPSIZES
C      D--A FACTOR WHICH ALLOWS THE USER TO SELECT THE ACCURACY TO WHICH THE
C          BASE SOLUTION WILL BE CALCULATED
C      N--THE NUMBER OF SPATIAL STEPS
C      KCOUNT--THE NUMBER OF VALUES OF ALPHA
C      WNUM-- WAVE NUMBER
C      W--ANGULAR FREQUENCY
C      DISPLI--DISPLACEMENT AMPLITUDE
C      ALPH--A SPECIFIC VALUE OF ALPHA FOR INPUT TO KB
```

```
C     RKAPPA--INVERSE OF THE DISCONTINUITY DISTANCE
C MAIN REQUIRES AS INPUT BETA,EPS,DELX,DELT,D,N,PI,KCOUNT,AND THE VECTOR ALPHA
      READ 105,BETA,EPS,DELX,DELT,D,N,PI,KCOUNT
  105 FORMAT(F4.1,F6.4,2F5.3,F10.8,I5,F15.8,I5)
      PRINT 106
  106 FORMAT(2X,'BETA',3X,'EPS',3X,'DELX',3X,'DELT',7X,'D',8X,'N',11X,'P
     1I',7X,'KCOUNT')
      PRINT 107,BETA,EPS,DELX,DELT,D,N,PI,KCOUNT
  107 FORMAT(' ',F4.1,2X,F6.4,2X,F5.3,2X,F5.3,2X,F10.8,2X,I5,2X,F15.8,2X
     1,I7/)
      READ 110,(ALPHA(I),I=1,KCOUNT)
  110 FORMAT(8F10.8)
      PRINT 111
  111 FORMAT(' ALPHA(1) TO ALPHA(KCOUNT) RESPECTIVELY')
      PRINT 112,(ALPHA(I),I=1,KCOUNT)
  112 FORMAT(' ',6F13.8/)
      NM=N-1
      WNUM=2.*PI
      RKAPPA=BETA*EPS*WNUM
      W=2.*PI
      DISPLI=EPS/W
C CALCULATE FUBINI SOLUTION AT THE BOUNDRY,SET BOUNDRY VALUES
      K=2*N
      IC=4*N
      IDO=2*(N+1)
C  VARIABLES SUCH AS IDO AND JDO APPEAR THROUGHOUT AND ARE USED AS DO LOOP
C   PARAMETERS,IC IS GENERALLY USED AS A COUNTER FOR PROPER SUBSCRIPTING OF
C    ARRAYS,HERE IT IS A DO PARAMETER ALSO
      T=0.
      DO 28 I=IDO,IC
      WT=W*T
      FCOS=COS(WT)
      FSIN=SIN(WT)
      HCOS=COS(2.*WT)
      FI(I)=-DISPLI*(RKAPPA/4.*HCOS+WNUM*FSIN)
      S(I)=-DISPLI*(RKAPPA*WNUM*SIN(2.*WT)-WNUM*WNUM*FCOS)
```

```
      TH(I)=-DISPLI*(2.*RKAPPA*W*WNUM*FCOS+WNUM*WNUM*W*FSIN)
      FUB(I)=-DISPLI*FCOS
      FUB(I-K)=FUB(I)-DELX*FI(I)+DELX*DELX/2.*S(I)
   28 T=(I-IDO+1)*DELT
      VFUB(1)=0.
      ICOR=4*N+1
      ST=DELT
      X=DELX
      JC=3
      IC=4*N
      IDO=N+1
C  THE FOLLOWING LOOP CALCULATES THE BASE SOLUTION FOR THE REMAINDER OF THE
C   SOLUTION SPACE IN X AND T,JC IS THE NUMBER OF HARMONICS REQUIRED FOR THE
C    ACCURACY REQUIRED BY D
      DO 4 I=3,IDO
      JDO=2*(N-I+1)+1
      DO 6 J=1,JDO
    6 SUM(J)=0.
      VSUM=0.
      DO 8 NN=1,JC
      RKX=NN*RKAPPA*X
      WNX=WNUM*X
C  BESJ IS AN SSP ROUTINE WHICH COMPUTES J BESSEL FUNCTIONS
      CALL BESJ(RKX,NN,BJ,D,IER)
      IF (IER.NE.0) PRINT 115,X,NN,IER
  115 FORMAT('0','ERROR IN BESJ AT ',F6.2,' METERS,NN=',I3,' IER=',I2)
      VSUM=BJ*SIN(NN*(DELT*NM*W-WNX))/RKX+VSUM
      T=ST
      DO 10 J=1,JDO
      SUM(J)=BJ*COS(NN*(W*T-WNX))/RKX/NN+SUM(J)
   10 T=ST+J*DELT
    8 CONTINUE
      TEST=BJ/RKX/NN
      IF(TEST.GE.D) JC=JC+1
      DO 12 J=1,JDO
      IC=IC+1
```

```
   12 FUB(IC)=-DISPLI*SUM(J)*2.
      VFUB(I-1)=EPS*VSUM*2.
      X=(I-1)*DELX
    4 ST=(I-1)*DELT
C THIS BUFFER SELECTS VALUES OF FUB AT A FIXED TIME DETERMINED BY THE INITIAL
C   VALUE OF K
C THIS K SELECTS A TIME WHICH CORRESPONDS TO THE ALTITUDE OF THE TRIANGULAR
C   SOLUTION MESH
      K=3*N+1
      DO 66 I=2,N
      DIFFS(I-1)=FUB(K)
   66 K=K+2*(N-I+1)
C  VALUES OF DDEL AND ZDEL ARE COMPUTED HERE FOR EXTERNAL SCALING OF
C   PARTICLE DISPLACEMENT PLOTS
      X=DIFFS(1)
      YDEL=DIFFS(1)
      DO 14 I=2,NM
      YDEL=AMAX1(DIFFS(I),YDEL)
      X=AMIN1(DIFFS(I),X)
   14 CONTINUE
      ZDEL=AMAX1(ABS(YDEL),ABS(X))
C  THE DIVISOR HERE IS THE NUMBER OF LINES DESIRED IN THE PLOT
      DDEL=2.*ZDEL/31.
C  OUTPUT--VECTORS AND PLOTS OF SELECTED VALUES OF THE INVISCID SOLUTION
      CALL PLOT(DIFFS,NM,DDEL,ZDEL,1)
      PRINT 170
  170 FORMAT('0PLOTTED VALUES OF THE FUBINI SOLN')
      PRINT 125,(DIFFS(I),I=1,NM)
      CALL PLOT(VFUB,NM,DDEL,ZDEL,2)
      PRINT 444
  444 FORMAT('0PLOTTED VALUES OF THE FUBINI SOLN FOR VELOCITY')
      PRINT 125 ,(VFUB(I),I=1,NM)
C  THE PARAMETER SPACE EQUATION IS FIRST SOLVED ONCE FOR ALPHA=0 TO ESTABLISH
C   THE BASE VALUES OF THE PARAMETER SPACE VARIABLE,G
C  SET BOUNDARY VALUES FOR GOUT
      DO 16 J=1,ICOR
```

```
   16 GOUT(J)=0.
C CALCULATE VARIABLE COEFFICIENTS
      DO 20 J=3,N
      K=2*N+1
      DO 22 I=1,J
      IU=2*N-J+K
      Y(I)=FUB(K+J)
      YU(I)=FUB(IU)
   22 K=K+2*(N-I)
      IF (J.LT.5) GOTO 24
C CALCULATE FIRST AND SECOND PARTIALS WITH RESPECT TO X ,DET3 AND DET5 ARE SSP
C   ROUTINES ,DET3 IS REQUIRED NEAR THE CORNERS OF THE SOLUTION TRIANGLE
C    BECAUSE THERE ARE TOO FEW POINT FOR DET5 AT THESE POSITIONS
      CALL DET5(DELX,Y,Z,J,IER)
      CALL DET5(DELX,Z,V,J,IER)
      CALL DET5(DELX,YU,ZU,J,IER)
      CALL DET5(DELX,ZU,VU,J,IER)
      GOTO 26
   24 CALL DET3(DELX,Y,Z,J,IER)
      CALL DET3(DELX,Z,V,J,IER)
      CALL DET3(DELX,YU,ZU,J,IER)
      CALL DET3(DELX,ZU,VU,J,IER)
   26 K=4*N-1
      DO 20 I=2,J
      IU=2*N-J+K
      FI(J+K)=Z(I)
      S(J+K)=V(I)
      FI(IU)=ZU(I)
      S(IU)=VU(I)
   20 K=K+2*(N-I)
C  X-DERIVATIVES AT THE CORNERS ARE COMPUTED BASED ON TWO POINTS ONLY
      IUCOR=6*N-3
      FI(ICOR)=(FUB(ICOR)-FUB(2*N+3))/DELX
      S(ICOR)=(FUB(ICOR)-2.*FUB(2*N+3)+FUB(3))/DELX/DELX
      FI(IUCOR)=(FUB(IUCOR)-FUB(4*N-1))/DELX
      S(IUCOR)=(FUB(IUCOR)-2.*FUB(4*N-1)+FUB(2*N-1))/DELX/DELX
```

```
      IC=4*N
      IDO=N-1
      DO 30 I=3,IDO
      JDO=2*(N-I+1)+1
      DO 32 J=1,JDO
      IC=IC+1
   32 Y(J)=S(IC)
      IC=IC-JDO
C  THE SPACE TIME DERIVATIVES TH ARE CALCULATED WITH ROUTINE DET5
      CALL DET5(DELT,Y,U,JDO,IER)
      DO 30 J=1,JDO
      IC=IC+1
   30 TH(IC)=U(J)
      IC=2*N+1
      PRINT 140
  140 FORMAT('0',6X,'F',11X,'C00',10X,'C01',10X,'C10',10X,'C11',10X,'C12
     1',10X,'C20',10X,'C21')
      DO 36 I=2,IDO
C  IC IS A COUNTER,NIC STEPS BACK ONE STEPSIZE IN SPACE,ICP ONE STEPSIZE AHEAD
      IC=IC+3
      JDO=2*(N-I+1)
      DO 34 J=3,JDO
      NIC=IC-JDO-2
      ICP=IC+JDO
C  THE PARAMETER SPACE EQUATION VARIABLE COEFFICIENTS ARE COMPUTED BASED ON THE
C   VALUES OF THE DERIVATIVES AT THE CENTER OF THE SOLUTION CUBE
      DUM=(1.+FI(IC))**9
      A1=1.+FI(IC)
      A2=DUM
      A3=2.*BETA*S(IC)
      A4=DUM*ALPHA(1)
      F=-DUM*TH(IC)
C  THE C00,C01,ETC. TERMS ARE THE COEFFICIENTS OF THE VARIOUS POINTS IN THE
C   SOLUTION CUBE, C02,C22 ARE MISSING BECAUSE THESE POINTS ARE NOT REQUIRED IN
C    THIS FINITE DIFFERENCE SCHEME
      C00=-A4/DELX/DELX/DELT
```

```
      C01=(A1/DELX+A3/2.+A4/DELX/DELT)/DELX
      C10=-A2/DELT/DELT-2.*C00
      C11=-2.*((A1+A4/DELT)/DELX/DELX-A2/DELT/DELT)
      C12=-A2/DELT/DELT
      C20=C00
      C21=C01-A3/DELX
      IF(IC.NE.(N+1)*(1+3*N)/4) GOTO 98
C  THE FOLLOWING OUTPUT IS GENERATED AS A SPOT CHECK ON THE BEHAVIOR OF G
C   AND THE VARIABLE COEFFICIENTS OF THE SOLUTION CUBE
      K=1
      ICOUNT =0
      PRINT 130,K,ICOUNT,I,J
  130 FORMAT('0','K=',I2,' ICOUNT=',I2,'I=',I4,' J=',I4)
      PRINT 135 ,F,C00,C01,C10,C11,C12,C20,C21
  135 FORMAT(' ',8E13.4)
      PRINT 160,GOUT(NIC-1),GOUT(NIC),GOUT(IC-1),GOUT(IC),GOUT(IC+1),GOU
     1T(ICP-1)
C THIS STEP SOLVES THE FINITE DIFFERENCE CUBE
   98 GOUT(ICP)=(F-C00*GOUT(NIC-1)-C01*GOUT(NIC)-C10*GOUT(IC-1)-C11*GOUT
     1(IC)-C12*GOUT(IC+1)-C20*GOUT(ICP-1))/C21
   34 IC=IC+1
      KG=JDO-2
C  THIS STATEMENT ESTIMATES THE VALUE OF GOUT AT THE BASE OF THE SUBSEQUENT
C   COLUMN OF GOUT VALUES IN ORDER TO ALLOW EXPLICIT CALCULATION OF THE
C    REMAINDER OF THE COLUMN
C
   36 GOUT(IC+2+KG)=2.5*GOUT(IC+2)-2.*GOUT(IC-KG)+.5*GOUT(IC-2*KG-4)
C  THE FOLLOWING LOOP REPEATS THE ABOVE COMPUTATIONS FOR SUCCESSIVE VALUES OF
C   ALPHA AND PERFORMS THE QUADRATURE WHICH RETURNS THE DISPLACEMENT SOLUTION
C    AT EACH ALPHA STEP
      DO 38 K=2,KCOUNT
      JDO=N*(N+2)
C  GOUT IS RENAMED GIN TO PREPARE FOR A NEW CALCULATION OF GOUT
      DO 39 J=1,JDO
   39 GIN(J)=GOUT(J)
      ICOUNT =0
```

```
      IC=ICOR
      DO 40 I=3,N
      JDO=2*(N-I+1)+1
C  AN INITIAL ESTIMATE OF THE INCREMENT TO BE APPLIED TO THE DISPLACEMENT
C   SOLUTION IS MADE BASED ON GIN
      DO 40 J=1,JDO
      DELE=(ALPHA(K)-ALPHA(K-1))*GIN(IC)
      FUB(IC)=FUB(IC)+DELE
   40 IC=IC+1
C  COEFFICIENTS ARE CALCULATED AS BEFORE FOR ICOUNT=0 BASED ON THE INITIAL
C   ESTIMATE OF THE DISPLACEMENT SOLUTION,FOR ICOUNT =1,THE COEFFICIENTS ARE
C    CALCULATED BASED ON THE SOLUTION AS IT IS SUBSEQUENTLY MODIFIED
   60 DO 42 J=3,N
      M=2*N+1
      DO 44 I=1,J
      Y(I)=FUB(J+M)
      IU=2*N-J+M
      YU(I)=FUB(IU)
   44 M=M+2*(N-I)
      IF (J.LT.5) GOTO 46
      CALL DET5(DELX,Y,Z,J,IER)
      CALL DET5(DELX,Z,V,J,IER)
      CALL DET5(DELX,YU,ZU,J,IER)
      CALL DET5(DELX,ZU,VU,J,IER)
      GO TO 48
   46 CALL DET3(DELX,Y,Z,J,IER)
      CALL DET3(DELX,Z,V,J,IER)
      CALL DET3(DELX,YU,ZU,J,IER)
      CALL DET3(DELX,ZU,VU,J,IER)
   48 M=4*N-1
      DO 42 I=2,J
      IU=2*N-J+M
      FI(J+M)=Z(I)
      FI(IU)=ZU(I)
      S(J+M)=V(I)
      S(IU)=VU(I)
```

```
   42 M=M+2*(N-I)
      FI(ICOR)=(FUB(ICOR)-FUB(2*N+3))/DELX
      S(ICOR)=(FUB(ICOR)-2.*FUB(2*N+3)+FUB(3))/DELX/DELX
      FI(IUCOR)=(FUB(IUCOR)-FUB(4*N-1))/DELX
      S(IUCOR)=(FUB(IUCOR)-2.*FUB(4*N-1)+FUB(2*N-1))/DELX/DELX
      IC=4*N
      IDO=N-1
      DO 52 I=3,IDO
      JDO=2*(N-I+1)+1
      DO 54 J=1,JDO
      IC=IC+1
   54 Y(J)=S(IC)
      IC=IC-JDO
      CALL DET5(DELT,Y,U,JDO,IER)
      DO 52 J=1,JDO
      IC =IC+1
   52 TH(IC)=U(J)
C  THE SOLUTION TECHNIQUE FOR GOUT IS IDENTICAL TO THAT FOR ALPHA=0
      IC=2*N+1
      DO 56 I=2,IDO
      IC=IC+3
      JDO=2*(N-I+1)
      DO 58 J=3,JDO
      NIC=IC-JDO-2
      ICP=IC+JDO
      DUM=(1.+FI(IC))**9
      A1=1.+FI(IC)
      A2=DUM
      A3=2.*BETA*S(IC)
      A4=DUM*ALPHA(K)
      F=-DUM*TH(IC)
      C00=-A4/DELX/DELX/DELT
      C01=(A1/DELX+A3/2.+A4/DELX/DELT)/DELX
      C10=-A2/DELT/DELT-2.*C00
      C11=-2.*((A1+A4/DELT)/DELX/DELX-A2/DELT/DELT)
      C12=-A2/DELT/DELT
```

```
      C20=C00
      C21=C01-A3/DELX
      IF(IC.NE.(N+1)*(1+3*N)/4) GOTO 99
      PRINT 130,K,ICOUNT,I,J
      PRINT 135,F,C00,C01,C10,C11,C12,C20,C21
      PRINT 160,GOUT(NIC-1),GOUT(NIC),GOUT(IC-1),GOUT(IC),GOUT(IC+1),GOU
     1T(ICP-1)
  160 FORMAT(' ','VALUES OF G  ',6E13.4)
   99 GOUT(ICP)=(F-C00*GOUT(NIC-1)-C01*GOUT(NIC)-C10*GOUT(IC-1)-C11*GOUT
     1(IC)-C12*GOUT(IC+1)-C20*GOUT(ICP-1))/C21
      IF(ICOUNT.EQ.1)GOTO 58
C  THE FOLLOWING STEP MODIFIES THE INITIAL ESTIMATE OF THE INCREMENT TO BE
C   APPLIED TO THE DISPLACEMENT SOLUTION RESULTING IN WHAT AMOUNTS TO A
C    TRAPEZOIDAL INTEGRATION; WHEN ICOUNT=1,THIS STEP IS BYPASSED AND AFTER THE
C     LOOPS ARE COMPLETED A NEW VALUE OF ALPHA IS PICKED UP
      DELE=(GOUT(ICP)-GIN(ICP))/2.*(ALPHA(K)-ALPHA(K-1))
      FUB(ICP)=FUB(ICP)+DELE
   58 IC=IC+1
      KG=JD0-2
   56 GOUT(IC+2+KG)=2.5*GOUT(IC+2)-2.*GOUT(IC-KG)+.5*GOUT(IC-2*KG-4)
      ICOUNT=ICOUNT+1
      IF (K.EQ.KCOUNT) GOTO 238
      IF(ICOUNT.EQ.1) GOTO 60
C  THESE STATEMENTS PICK VALUES OF ALPHA FOR WHICH OUTPUT IS DESIRED
      IF(K.EQ.5) GOTO 238
      IF(K.EQ.8)GOTO 238
      IF(K.EQ.12) GOTO 238
      GOTO 38
C  THIS LOOP SELECTS AN OUTPUT VECTOR AT A FIXED TIME DETERMINED BY M,COMPUTES
C   THE DIFFERENCE BETWEEN THIS VECTOR AND DIFFS WHICH CONTAINS VALUES OF THE
C    INVISCID SOLUTION,AND ALSO COMPUTES THE PARTICLE VELOCITY AT THE SAME FIXED
C     TIME
  238 M=3*N+1
      DO 62 I=2,N
      OUT(I-1)=FUB(M)
      DIFF(I-1)=DIFFS(I-1)-FUB(M)
```

```
      IF(I.EQ.N) GOTO 69
      PV(I-1)=(FUB(M-2)-8.*(FUB(M-1)-FUB(M+1))-FUB(M+2))/12./DELT
      GOTO 62
   69 PV(I-1)=(FUB(M+1)-FUB(M-1))/2./DELT
   62 M=M+2*(N-I+1)
      ALPH=ALPHA(K)
      GAMMA=BETA*EPS/PI/ALPH
      PRINT 555,GAMMA
  555 FORMAT('0THE RATIO OF 1/ALPHA TO 1/BEK FOR THIS PROBLEM IS',F10.2)
C  OUTPUT-- VECTORS AND PLOTS OF
C       1) KECK-BEYER SOLUTION FOR DISPLACEMENT AND VELOCITY
C       2) PARAMETRIC DIFFERENTIATION SOLUTION FOR DISPLACEMENT AND VELOCITY
C       3) THE DIFFERENCES BETWEEN THE INVISCID SOLUTION AND THE TWO VISCOUS
C            SOLUTIONS
      CALL KB(DIFFS,ALPH,RKAPPA,WNUM,DELX,DELT,N,PI,DISPLI,DDEL,ZDEL)
      CALL PLOT(OUT,NM,DDEL,ZDEL,1)
      PRINT 145
  145 FORMAT('0','PLOTTED VALUES OF CALCULATED SOLUTION')
      PRINT 125,(OUT(I),I=1,NM)
      CALL PLOT (PV,NM,DDEL,ZDEL,2)
      PRINT 165
  165 FORMAT('0PLOTTED VALUES OF PARTICLE VELOCITY')
      PRINT 125,(PV(I),I=1,NM)
      CALL PLOT(DIFF,NM,DDEL,ZDEL,2)
      PRINT 153
  153 FORMAT('0','PLOTTED VALUES OF THE DELTA BETWEEN INVISCID AND VISCO
     1US SOLUTIONS')
      PRINT 125,(DIFF(I),I=1,NM)
  125 FORMAT(' ',8E16.7)
   38 CONTINUE
      STOP
      END
```

```
      SUBROUTINE KB(FUBGH,ALPH,RKAPPA,WNUM,DELX,DELT,N,PI,DISPLI,DDEL,Z
     1DEL)
C   KB CALCULATES THE KECK - BEYER SOLUTION IN TERMS OF PARTICLE DISPLACEMENT AND
C    VELOCITY AND COMPUTES THE DIFFERENCE BETWEEN THE INVISCID AND THE KB
C     DISPLACEMENT SOLUTIONS
C   KB INPUT VARIABLES ARE ALL COMMON WITH MAIN VARIABLES
C      FUBGH-- AN INPUT VECTOR OF VALUES OF THE INVISCID SOLUTION
C   KB INTERNAL VARIABLES
C      COEFF1,2,3,4,5,6-- COEFFICIENTS OF THE KECK BEYER SOLUTION
C      BKDIF-- A VECTOR OF DIFFERENCE VALUES
C      BEYER-- THE VECTOR OF KB DISPLACEMENT VALUES
C      VBEYER-- THE VECTOR OF KB VELOCITY VALUES
C      ATTEN-- SMALL SIGNAL ATTENUATION COEFFICIENT
C      EX1,2,3,4,5,6-- EXPONENTIAL DECAY TERMS
      DIMENSION FUBGH(105),BKDIF(105),BEYER(105),VBEYER(105)
      IDO=N-1
      ATTEN=PI*WNUM*ALPH
      DO 2 I=1,IDO
    2 BKDIF(I)=FUBGH(I)
C CALCULATE K-B SOLN
      X=0.
      DO 6 I=1,IDO
      EX1=EXP(-ATTEN*X)
      EX2=EX1*EX1
      EX3=EX2*EX1
      EX4=EX3*EX1
      EX5=EX4*EX1
      EX6=EX5*EX1
      WNX=WNUM*(DELT*IDO-X)
      RK=RKAPPA/4./ATTEN*(1.-EX2)
      COEFF1=1.-RK**2/2.+RK**4/12.*(4.-EX2*(2.+EX2))-RK**6/144.*(33.-EX2
     1*(54.+EX2*(18.-EX2*(18.+EX2*(15.+EX2*(6.+EX2))))))
      COEFF2=1.-RK**2/3.*(3.+EX2)+RK**4/24.*(22.+EX2*(3.-EX2*(5.+EX2*(3.
     1+EX2))))
      COEFF3=2.+EX2-RK**2/4.*(12.+EX2*(10.+EX2*(4.+EX2)))+RK**4/40.*(140
     1.+EX2*(105.+EX2*(30.-EX2*(4.+EX2*(14.+EX2*(9.+EX2*(4.+EX2)))))))
```

```
      COEFF4=(6.+EX2*(6.+EX2*(3.+EX2)))-RK**2/5.*(60.+EX2*(80.+EX2*(60.+
     1EX2*(35.+EX2*(15.+EX2*(5.+EX2))))))
      COEFF5=(24.+EX2*(36.+EX2*(30.+EX2*(20.+EX2*(10.+EX2*(4.+EX2))))))-
     1RK**2/6.*(360.+EX2*(600.+EX2*(690.+EX2*(570.+EX2*(395.+EX2*(240.+E
     2X2*(126.+EX2*(56.+EX2*(21.+EX2*(6.+EX2))))))))))
      COEFF6=RK**5/120.*(120.+EX2*(240.+EX2*(270.+EX2*(240.+EX2*(180.+EX
     12*(120.+EX2*(70.+EX2*(35.+EX2*(15.+EX2*(5.+EX2))))))))))
      BEYER(I  )=-DISPLI*(EX1*COS(WNX)*COEFF1+EX2*COS(2.*WNX)*COEFF2*RK/
     12.+EX3*COS(3.*WNX)*COEFF3*RK**2/6.+EX4*COS(4.*WNX)*COEFF4*RK**3/24
     2.+EX5*COS(5.*WNX)*COEFF5*RK**4/120.+EX6*COS(6.*WNX)*COEFF6/6.)
      VBEYER(I)=DISPLI*WNUM*(EX1*SIN(WNX)*COEFF1+EX2*SIN(2.*WNX)*COEFF2*
     1RK+EX3*SIN(3.*WNX)*COEFF3*RK**2/2.+EX4*SIN(4.*WNX)*COEFF4*RK**3/6.
     2+EX5*SIN(5.*WNX)*COEFF5*RK**4/24.+EX6*SIN(6.*WNX)*COEFF6)
    6 X=I*DELX
C CALCULATE THE DELTA BETWEEN SOLNS
      DO 4 I=1,IDO
    4 BKDIF(I)=BKDIF(I)-BEYER(I)
      CALL PLOT(BEYER,IDO,DDEL,ZDEL,1)
      PRINT 100
  100 FORMAT('0PLOTTED VALUES OF THE KECK-BEYER SOLN')
      PRINT 101,(BEYER(I),I=1,IDO)
  101 FORMAT(' ',8E16.7)
      CALL PLOT(BKDIF,IDO,DDEL,ZDEL,2)
      PRINT 105
  105 FORMAT('0PLOTTED VALUES OF THE DIFFERENCE BETWEEN KB AND FUBINI')
      PRINT 101,(BKDIF(I),I=1,IDO)
      CALL PLOT(VBEYER,IDO,DDEL,ZDEL,2)
      PRINT 110
  110 FORMAT('0PLOTTED VALUES OF K/B VELOCITY')
      PRINT 101,(VBEYER(I),I=1,IDO)
      RETURN
      END
```

```
      SUBROUTINE PLOT(DAT,NOUT,DDEL,ZDEL,ISCALE)
C  PLOT PRODUCES ON LINE PLOTS OF EQUALLY SPACED DATA FOR UP TO 129 POINTS
C   PLOT ALLOWS INTERNAL OR EXTERNAL SCALING OF THE ORDINATE,THE ABCISSA IS NOT
C    SCALED
C  PLOT INPUT VARIABLES
C      DAT-- A VECTOR OF VALUES TO BE PLOTTED
C      NOUT-- THE NUMBER OF POINTS IN DAT
C      DDEL-- RANGE OF ONE QUANTIZED PLOTTING INTERVAL
C      ZDEL-- MAXIMUM EXTENT OF THE ORDINATE VALUES,POSITIVE AND NEGATIVE
C      ISCALE-- ISCALE DETERMINES WHETHER PLOT USES INTERNAL OR EXTERNAL
C                SCALING
C  WITH EXTERNAL SCALING DDEL AND ZDEL ARE INPUT VARIABLES
C  WITH INTERNAL SCALING PLOT SCANS THE DATA AND DETERMINES AN ABSOLUTE
C   MAXIMUM TO WHICH ZDEL IS SET,DDEL IS THEN DETERMINED BY ZDEL AND THE
C    NUMBER OF LINES DESIRED IN THE PLOT
C ISCALE=1 IMPLIES EXTERNAL SCALING
C ISCALE=ANY OTHER INTEGER IMPLIES INTERNAL SCALING
C
      DIMENSION DAT(130),IOUT(130)
      DATA IBL/1H /,IAST/1H*/,IOR/1HO/
      IF (ISCALE.EQ.1) GOTO 11
      X=DAT(1)
      Y=DAT(1)
      DO 10 I=2,NOUT
      Y=AMAX1(DAT(I),Y)
      X=AMIN1(DAT(I),X)
   10 CONTINUE
      Z=AMAX1(ABS(Y),ABS(X))
C  THE DIVISOR HERE IS THE NUMBER OF LINES DESIRED IN THE PLOT
      DEL=2.*Z/31.
      GOTO 12
   11 DEL=DDEL
      Z=ZDEL
   12 PRINT 20
   20 FORMAT(1H1)
      PRINT 25,DEL
```

```
   25 FORMAT('0QUANTIZED SPACING INTERVAL =',E13.4/)
C     THE SECOND DO PARAMETER IS THE NUMBER OF LINES DESIRED IN THE PLOT
      DO 40 L=1,31
      DO 30 I=1,NOUT
      IOUT(I)=IBL
      IF((DAT(I).LE.(Z-((L-1)*DEL))).AND.(DAT(I).GE.(Z-L*DEL))) GOTO 60
      GOTO 30
   60 IOUT(I)=IAST
      IF(L.EQ.16) IOUT(I)=IOR
   30 CONTINUE
      PRINT 70,(IOUT(I),I=1,NOUT)
   70 FORMAT(' ',129A1)
   40 CONTINUE
      RETURN
      END
```

Thesis
Z2475 Zavadil
165000
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